A TREATISE

ON THE

DIFFERENTIAL CALCULUS;

WITH ITS APPLICATION TO

PLANE CURVES, TO CURVE SURFACES,

AND TO

CURVES OF DOUBLE CURVATURE.

BY

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TO THE

VERY REV. ROBERT HALDANE, D.D., F.R.S.E.

PRINCIPAL OF ST MARY'S COLLEGE, ST ANDREWS,

AND TO

THOMAS DUNCAN, ESQ. M.A.

PROFESSOR OF MATHEMATICS IN THE UNITED COLLEGE, ST ANDREWS

The Following Treatise

IS RESPECTFULLY INSCRIBED BY

THEIR MOST OBEDIENT AND OBLIGED SERVANT,

THE AUTHOR.

PREFACE.

The Author has for many years taught an extensive course of Mathematics; and having learned from experience how difficult it is to excite a general taste for the higher branches of analysis among the rising generation, he has been induced to write the following Treatise.

He conceived that if the fundamental principles of the Differential Calculus were explained in a simple manner, and a sufficient number of well-selected examples given on each chapter, many who are repelled by the difficulties which occur at the commencement of the subject, would be induced not only to enter on the study, but even to master the higher and more abstruse branches of the science.

With this view he has chosen the method of limits in preference to that of derived functions, as it is easier, in his opinion, to find the value of a ratio whose terms are evanescent, than to establish Taylor's Theorem by the operations of common Algebra alone, and then to develope by the same means such functions as log. $(x + h)$, sin. $(x + h)$, &c. in ascending powers of h.

It yields the Author great pleasure to find that he is not singular in his opinion: as in most of the treatises on the Differential Calculus which have been published of late years, both in this country and in France, the method of limits is followed.

As it is desirable, however, that the student should be made acquainted with the different methods that have been adopted for deducing the principles of differentiation, the Author has fully explained them in his 18th chapter, and applied them to prove several propositions.

The Author has been particularly careful in his examples to advance from the simpler to the more abstruse; and while he has given full solutions of a considerable number on each chapter, he has merely given the answers of the remainder, as nothing is better fitted to excite the taste of the student than the pleasure of solving problems by his own unaided exertions.

The Author has attempted to simplify many of the demonstrations; and some of them, as far as he knows, are entirely his own. He has chosen the symbol $\frac{dz}{dx}$ to represent the first differential co-efficient of z, considered as a function of x, in preference to $d_x z$, because the latter has never been generally adopted, even in that University where it was first introduced, and because it would still be necessary for the student to accustom himself to the notation of Leibnitz before he could read the works of Biot, Poisson, Lacroix, and Laplace, to say nothing of those of Airy, Whewell, and Pratt.

If this Treatise shall contribute in any degree to advance the study of a most interesting and important branch of analysis, the Author will feel himself amply rewarded for the trouble he has taken in its preparation.

In conclusion, the Author has great pleasure in thus publicly returning his best thanks to his excellent friend and preceptor, Professor Duncan of St Andrews, who examined this work in manuscript, and suggested several improvements, which were adopted.

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DIFFERENTIAL CALCULUS.

CHAPTER I.

DEFINITIONS.

(1.) QUANTITIES are divided into constant and variable.

(2.) *Constant quantities* retain the same values throughout any calculation in which they are employed. They are represented by the first letters of the alphabet, a, b, c, &c.

(3.) *Variable quantities* are those to which different values may be assigned in the course of the same calculation. They are *usually* represented by the last letters of the alphabet, x, y, z, &c.

(4.) A quantity is said to be a *function* of another when it is equal to any algebraical expression of the other. Thus, in the equations $z = ax + b$, $z = (a + x)^n$, $z = a^x$, $z = \log. (x)$, $z = \sin. (a + x)$. z is said to be functions of x. Functions of x are in general represented thus: $z = f(x)$, $z = \text{F}(x)$, $z = \varphi(x)$, &c.

(5.) An *explicit function* is one where z is known in terms of x, as $z = ax^2 + bx + c$.

(6.) An *implicit function* is one where x and z are involved together; thus, $az^2 + bx^3 - cz = 0$ is an implicit function of x. It is written thus: $f(x, z) = 0$, or $\text{F}(x, z) = 0$.

(7.) A *transcendental function* is either exponential, logarithmic, or trigonometrical, as $z = a^x$, $z = \log. (a + x)$, $z = \tan. (a - x)$.

(8.) All functions which are not transcendental are called *algebraical functions*, as $z = a^3 + x^3$, $z = \sqrt{a^2 - x^2}$, $z = (a + x)^m$.

(9.) If the relation between x and z be represented by an equation

of the form $z = f(x)$, x is called the *independent variable* and z the *dependent variable.*

(10.) The *increment* or *decrement* of any function is the difference of two particular values of it corresponding to different values of the independent variable. Thus, let

$z = ax^2 + bx + c$: and when x becomes $= x'$ let z become $= z'$, then $z' = ax'^2 + bx' + c$

$$\therefore \quad z' \sim z = a(x'^2 \sim x^2) + b(x' \sim x)$$

$$\frac{z' \sim z}{x' \sim x} = a(x' + x) + b.$$

(11.) When the *limit* of the ratio of the simultaneous increments or decrements of the function and the independent variable is taken, it is expressed thus $\frac{dz}{dx} = 2\,ax + b.$

(12.) The object of the Differential Calculus is to find the limit of the ratio of the simultaneous increments or decrements of the function, and the variable on which it depends.

(13.) dz, d^2z, d^3z, . . . d^nz, are the first, second, third . . . and n^{th}, differentials of z, while dz^2, dz^3, . . . dz^n are the square cube and n^{th} powers of the first differential of the same quantity.

(14.) $\frac{dz}{dx}$ is called the first *differential coefficient* of z, considered as a function of x, because it is the multiplier of dx in the expression for dz. Thus, $\frac{dz}{dx} = 2\,ax + b$. (11.) $\therefore dz = (2\,ax + b)\,dx$.

Differentiation of Algebraical Functions of One Variable.

(15.) Let $z = u$ where u is a function of x, then $dz = du$.

For let x become equal to $x + h$, $u = u + k$ and $z = z'$, then $z' = u + k \; \therefore \frac{z' - z}{h} = \frac{k}{h}$. Taking the limits of both sides we have $\frac{dz}{dx} = \frac{du}{dx} \; \therefore dz = du$.

(16.) Let $z = u \pm a$ where u is a function of x, then $dz = du$.

For let $x = x + h$, $u = u + k$, and $z = z'$, then $z' = u \pm a + k$ $\therefore$ $\frac{z' - z}{h} = \frac{k}{h}$. Taking the limits of both sides, we have $\frac{dz}{dx} = \frac{du}{dx}$ $\therefore$ $dz = du$. Hence the differential of a variable function is equal to the differential of the same function increased or diminished by a constant quantity.

(17.) Let $z = au$ where u is a function of x, then $dz = adu$.

For let x become equal to $x + h$, $u = u + k$, and $z = z'$, then $z' = au + ak$ $\therefore$ $\frac{z' - z}{h} = a\frac{k}{h}$. Taking the limits of both sides we have $\frac{dz}{dx} = a\frac{du}{dx}$ $\therefore$ $dz = adu$. Hence the differential of the product of a variable function and a constant quantity is equal to the differential of the function multiplied by the constant quantity.

(18.) Let $z = u + v - w$ where u, v and w are functions of x, then $dz = du + dv - dw$.

For let x become equal to $x + h$, $u = u + k$, $v = v + l$, $w = w + m$, and $z = z'$, then $z' = u + k + v + l - w - m$ $\therefore$ $\frac{z' - z}{h} = \frac{k}{h} + \frac{l}{h} - \frac{m}{h}$. Taking the limits of both sides, we have $\frac{dz}{dx} = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}$ $\therefore$ $dz = du + dv - dw$. Hence the differential of any number of functions of the same variable, connected by the signs plus and minus, is equal to their differentials connected by the same signs.

(19.) Let $z = uv$ where u and v are functions of x, then $dz = vdu + udv$.

For let x become equal to $x + h$, $u = u + k$, $v = v + l$, and $z = z'$. then $z' = (u + k)(v + l) = uv + vk + ul + kl$ $\therefore$ $\frac{z' - z}{h} = v\frac{k}{h} + u\frac{l}{h} + \frac{kl}{h}$. Taking the limits of both sides, we have $\frac{dz}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$ $\therefore$ $dz = vdu + udv$. Hence the differential of the product of two functions of the same variable is equal to the differential of the

first multiplied by the second, plus the differential of the second multiplied by the first.

(20.) Let $z = uvw$ where u, v and w are functions of x, then $dz = vwdu + uwdv + uvdw$.

For let $vw = y$, then $dz = ydu + udy$ (19.) $= vwdu + u(wdv + vdw) = vwdu + uwdv + uvdw$.

(21.) Let $z = \frac{u}{v}$, where u and v are functions of x, then $dz = \frac{vdu - udv}{v^2}$.

For since $z = \frac{u}{v}$, $vz = u$ $\therefore$ $zdv + vdz = du$ $\therefore$ $vdz = du - zdv$ $= du - \frac{u}{v}dv = \frac{vdu - udv}{v}$ $\therefore$ $dz = \frac{vdu - udv}{v^2}$.

Hence the differential of the quotient of two functions of the same variable is equal to the differential of the numerator multiplied by the denominator, minus the differential of the denominator multiplied by the numerator, divided by the square of the denominator.

Cor. If $z = \frac{a}{v}$ then $dz = \frac{vda - adv}{v^2} = - \frac{adv}{v^2}$, since $da = 0$.

(22.) Let $z = u^n$ where u is a function of x, then $dz = nu^{n-1}du$.

For let $x = x + h$, $u = u + k$, and $z = z'$, then $z' = (u + k)^n$ $= u^n + nu^{n-1}k + \frac{n\overline{n-1}}{1 . 2}u^{n-2}k^2 + \dots$ $\therefore$ $\frac{z' - z}{h} = nu^{n-1}\frac{k}{h} + \frac{n\overline{n-1}}{1 . 2}u^{n-2}\frac{k^2}{h} + \dots$ Taking the limits of both sides we have $\frac{dz}{dx}$ $= nu^{n-1}\frac{du}{dx}$ $\therefore$ $dz = nu^{n-1}du$.

Hence the differential of any power of a function of a variable is

found by making the index the co-efficient, diminishing the index of the function by unity, and multiplying this product by the differential of the function.

(23.) Let z = a function of u, and u = a function of x, then $\frac{dz}{dx} = \frac{dz}{du} \cdot \frac{du}{dx}$.

For let $x = x + dx$, $u = u + du$, and $z = z + dz$, then since u is a function of x, we have $u + du = u + \frac{du}{dx} dx$. (14). Again, since z is a function of u, we have $z + dz = z + \frac{dz}{du} du = z + \frac{dz}{du} \cdot \frac{du}{dx} dx \therefore \frac{dz}{dx} = \frac{dz}{du} \cdot \frac{du}{dx}$.

COR.—If $z = x$ then $\frac{dx}{du} \cdot \frac{du}{dx} = 1 \therefore \frac{du}{dx} = \frac{1}{\frac{dx}{du}}$.

EXAMPLE (1.) Let $z = ax^{\frac{3}{2}} + bx + cx^{\frac{1}{2}} \therefore dz = \left(\frac{3}{2} ax^{\frac{1}{2}} + b + \frac{1}{2} cx^{-\frac{1}{2}}\right) dx$, and $\frac{dz}{dx} = \frac{3\, ax^{\frac{1}{2}}}{2} + b + \frac{c}{2\, x^{\frac{1}{2}}}$.

Ex. (2) Let $z = (a^2 + x^2)^n \therefore dz = n\, (a^2 + x^2)^{n-1}\, 2\, xdx$ $\therefore \frac{dz}{dx} = 2\, nx\, (a^2 + x^2)^{n-1}$.

Ex. (3.) Let $z = \sqrt{a + bx + cx^2} \therefore dz = \frac{1}{2}\, (a + bx + cx^2)^{-\frac{1}{2}}\, (b + 2\, cx)\, dx \therefore \frac{dz}{dx} = \frac{b + 2\, cx}{2\, (a + bx + cx^2)^{\frac{1}{2}}}$.

Ex. (4.) Let $z = (a^2 + x^2)^2 - (b^2 - x^2)^2 \therefore dz = 2\, (a^2 + x^2)\, 2\, xdx + 2\, (b^2 - x^2)\, 2\, xdx \therefore \frac{dz}{dx} = 4\, (a^2 + b^2)\, x$.

Ex. (5.) Let $z = \sqrt{\frac{a^2 + x^2}{a - x}} = \frac{(a^2 + x^2)^{\frac{1}{2}}}{(a - x)^{\frac{1}{2}}} \therefore dz =$

$$\frac{\frac{1}{2}\, (a + x^2)^{-\frac{1}{2}}\, (a - x)^{\frac{1}{2}}\, 2xdx - \frac{1}{2}\, (a - x)^{-\frac{1}{2}}\, (a^2 + x^2)^{\frac{1}{2}} \times -dx}{a - x}$$

$$= \frac{\dfrac{(a-x)^{\frac12} x\,dx}{(a^2+x^2)^{\frac12}} + \dfrac{(a^2+x^2)^{\frac12}\,dx}{2\,(a-x)^{\frac12}}}{a-x} = \frac{2\,(a-x)\;x\,dx + (a^2+x^2)\,dx}{2\;(a^2+x^2)^{\frac12}\;(a-x)^{\frac32}}$$

$$\therefore \frac{dz}{dx} = \frac{2\,ax - 2\,x^2 + a^2 + x^2}{2\;(a^2+x^2)^{\frac12}\;(a-x)^{\frac32}} = \frac{a^2 + 2\,ax - x^2}{2\,(a^2+x^2)^{\frac12}\,(a-x)^{\frac32}}$$

Ex. (6.) Let $z = \dfrac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} - \sqrt{1-x}} = \dfrac{(\sqrt{1+x} + \sqrt{1-x})^2}{2\,x}$

$$= \frac{1+\sqrt{1-x^2}}{x} \;\therefore\; dz = \frac{\frac12 (1-x^2)^{-\frac12}\,x \times -2\,x\,dx - (1+\sqrt{1-x^2})\,dx}{x^2}$$

$$= \frac{-\dfrac{x^2\,dx}{(1-x^2)^{\frac12}} - (1+\sqrt{1-x^2})\,dx}{x^2} = \frac{-x^2\,dx - (1-x^2)^{\frac12}dx - (1-x^2)\,dx}{x^2\,(1-x^2)^{\frac12}}$$

$$= -\frac{1+(1-x^2)^{\frac12}}{x^2\,(1-x^2)^{\frac12}}\,dx \;\therefore\; \frac{dz}{dx} = -\frac{1+(1-x^2)^{\frac12}}{x^2\,(1-x^2)^{\frac12}}.$$

Examples for Practice.

(1.) $z = ax^9 \;\therefore\; \dfrac{dz}{dx} = 9\,ax^8.$

(2.) $z = 4\,x^{\frac34} \;\therefore\; \dfrac{dz}{dx} = \dfrac{3}{x^{\frac14}}.$

(3.) $z = 7\,x^{-\frac23} \;\therefore\; \dfrac{dz}{dx} = -\dfrac{14}{3\,x^{\frac53}}.$

(4.) $z = 4\,ax^{-\frac12} \;\therefore\; \dfrac{dz}{dx} = -\dfrac{2\,a}{x^{\frac32}}.$

(5.) $z = x^3 + x^2 + x + 1 \;\therefore\; \dfrac{dz}{dx} = 3\,x^2 + 2\,x + 1.$

(6.) $z = (x^2 + x^3)^3 \;\therefore\; \dfrac{dz}{dx} = 3\,(x^2+x^3)^2\,(2\,x + 3\,x^2).$

(7.) $z = (a+3x)^8 \therefore \frac{dz}{dx} = 24(a+3x)^7.$

(8.) $z = (a^2 - 3x^2)^4 \therefore \frac{dz}{dx} = -24(a^2-3x^2)^3 x.$

(9.) $z = (bx^2 - x^4)^2 \therefore \frac{dz}{dx} = 4x(b-2x^2)(bx^2-x^4).$

(10.) $z = \frac{ax}{(b^2-x^2)^2} \therefore \frac{dz}{dx} = \frac{a(b^2+3x^2)}{(b^2-x^2)^3}.$

(11.) $z = \frac{(a+x)^2}{a^2-x^2} \therefore \frac{dz}{dx} = \frac{2a}{(a-x)^2}.$

(12.) $z = (a+x)\sqrt{a-x} \therefore \frac{dz}{dx} = \frac{a-3x}{2\sqrt{a-x}}.$

(13.) $z = \frac{x}{\sqrt{a-bx^2}} \therefore \frac{dz}{dx} = \frac{a}{(a-bx^2)^{\frac{3}{2}}}.$

(14.) $z = \frac{x^{\frac{3}{2}}}{\sqrt{a-x}} \therefore \frac{dz}{dx} = \frac{(3a-2x)x^{\frac{1}{2}}}{2(a-x)^{\frac{3}{2}}}.$

(15.) $z = \frac{x^4+4a^2x^2-8a^4}{\sqrt{a^2-x^2}} \therefore \frac{dz}{dx} = \frac{-3x^5}{(a^2-x^2)^{\frac{3}{2}}}.$

(16.) $z = \sqrt{x+\sqrt{1+x^2}} \therefore \frac{dz}{dx} = \frac{(x+\sqrt{1+x^2})^{\frac{1}{2}}}{2\sqrt{1+x^2}}.$

(17.) $z = \frac{x^4}{\sqrt{1+x^6}} \therefore \frac{dz}{dx} = \frac{4x^3+x^9}{(1+x^6)^{\frac{3}{2}}}.$

(18.) $z = \frac{(1+x^2)^{\frac{5}{2}}}{\sqrt{1-x}} \therefore \frac{dz}{dx} = \frac{(1+x^2)^{\frac{3}{2}}(1+10x-9x^2)}{2(1-x)^{\frac{3}{2}}}.$

(19.) $z = \dfrac{\sqrt{1 + x^2}}{\sqrt{1 + x}} \therefore \dfrac{dz}{dx} = \dfrac{x^2 + 2x - 1}{2(1 + x)^{\frac{3}{2}} (1 + x^2)^{\frac{1}{2}}}.$

(20.) $z = \dfrac{\sqrt{1 + x^2} + \sqrt{1 - x^2}}{\sqrt{1 + x^2} - \sqrt{1 - x^2}} \therefore \dfrac{dz}{dx} = -$
$\dfrac{2x}{\sqrt{1 - x^4}(1 - \sqrt{1 - x^4})}.$

CHAPTER II.

DIFFERENTIATION OF TRANSCENDENTAL FUNCTIONS OF ONE VARIABLE.

FIRST.—EXPONENTIAL AND LOGARITHMIC FUNCTIONS.

(24.) Let $z = a^x$ then $dz = A\,a^x\,dx$ where $A = (a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} + \&c.$

For let x become $= x + h$, then $z' = a^{x+h} = a^x \times a^h$.

Suppose $a = 1 + b$, then $a^h = (1+b)^h = 1 + h\,b + \frac{h\,\overline{h-1}}{1\,.\,2}b^2 + \frac{h\,\overline{h-1}\,\overline{h-2}}{1\,.\,2\,.\,3}b^3 + \&c.$ Arranging the terms according to the powers of h, it is easy to see that the two first terms of the development are $1 + \left(b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \&c.\right)h$. Let $A = b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \&c. = a - 1 - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} + \&c.$ and let B, C, D, &c. be the co-efficients of h^2, h^3, h^4, &c. then $a^h = 1 + A\,h + B\,h^2 + C\,h^3 + D\,h^4 + \&c.$ $\therefore z' = a^{x+h} = a^x \times a^h = a^x + A\,a^x\,h + a^x\,(B\,h^2 + C\,h^3 + D\,h^4 + \&c.)$ $\therefore \frac{z'-z}{h} = A\,a^x + a^x\,(B\,h + C\,h^2 + D\,h^3 + \&c.)$ Taking the limits of both sides we have $\frac{dz}{dx} = A\,a^x$ $\therefore dz = A\,a^x\,dx$.

(25.) To expand a^x in ascending powers of x.

$a^x = 1 + Ax + Bx^2 + Cx^3 + \&c.$ (24.) Differentiating with respect to x, we have $A\,a^x = A + 2\,B\,x + 3\,C\,x^2 + \&c.$ But $A\,a^x = A + A^2x + AB\,x^2 + AC\,x^3 + \&c.$ And when two series of the

above form are always equal, whatever be the value of x, their corresponding co-efficients are equal $\therefore$ B $= \frac{A^2}{2}$, C $= \frac{A^3}{1.2.3}$, &c. = &c. $\therefore$ $a^x = 1 + Ax + \frac{A^2 x^2}{1.2} + \frac{A^3 x^3}{1.2.3} +$ &c., which is the *exponential theorem.*

(26.) To find the value of A.

We have $a^x = 1 + Ax + \frac{A^2x^2}{1.2} + \frac{A^3x^3}{1.2.3} +$ &c.

Let $x = \frac{1}{A}$ then $a^{\frac{1}{A}} = 1 + \frac{1}{1.2} + \frac{1}{1.2.3} +$ &c. Calling the right-hand side of the equation e, we have $a^{\frac{1}{A}} = e \therefore a = e^A$ and log. a = A log. e $\therefore$ A $= \frac{\log. a}{\log. e}$. The number e, which is equal to $1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} +$ &c. = 2.7182818, is the base of the Napierian system of logarithms $\therefore$ log. $e = 1$ $\therefore$ A = log. a $\therefore$ da^x = log. $a\, a^x dx$.

N.B.—Whenever log. is employed, the Napierian logarithm is meant.

(27.) Let z = log. x, then $dz = \frac{dx}{x}$.

For since $z = \log x \therefore e^z = x$ and $\therefore Ae^z dz = dx$

$\therefore dz = \frac{dx}{Ae^z} = \frac{dx}{x}$, since A = 1 when the base is e.

(28.) Let z = log. y where y is a function of x, then $dz = \frac{dy}{y}$.

For $\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}(23) = \frac{1}{Ay} \frac{dy}{dx} = \frac{1}{y} \frac{dy}{dx}$.

Since A = 1 $\therefore dz = \frac{dy}{y}$.

Therefore the differential of the logarithm of any function is equal to the differential of the function divided by the function.

Ex. 1. Let $z = \frac{a^x - 1}{a^x + 1}$, then $dz = \frac{(a^x+1)\, d(a^x - 1) - (a^x - 1)\, d(a^x + 1)}{(a^x+1)^2}$

$= \frac{(a^x+1)\, a^x \log. a - (a^x - 1)\, a^x \log. a}{(a^x+1)^2} = \frac{2\, a^x \log. a}{(a^x+1)^2}\, dx \therefore \frac{dz}{dx} = \frac{2\, a^x \log. a}{(a^x+1)^2}$.

Ex. 2. Let $z = (x - 1)\, a^x \therefore dz = a^x\, dx + (x - 1)\, a^x \log. a\, dx = a^x\, dx + x\, a^x \log. a\, dx - a^x \log. a\, dx \therefore \frac{dz}{dx} = a^x\, x \log. a - (\log. a - 1)\, a^x$.

Ex. 3. Let $z = \log. \frac{\sqrt{1 + x}}{\sqrt{1 - x}} = \frac{1}{2} \log. (1 + x) - \frac{1}{2} \log. (1 - x)$

$\therefore dz = \frac{dx}{2\,(1 + x)} + \frac{dx}{2\,(1 - x)} = \frac{dx - x\, dx + dx + x\, dx}{2\,(1 - x^2)}$

$= \frac{dx}{1 - x^2} \therefore \frac{dz}{dx} = \frac{1}{1 - x^2}$.

Ex. 4. Let $z = \log. \left(\frac{\sqrt{a + x} + \sqrt{a - x}}{\sqrt{a + x} - \sqrt{a - x}}\right) = \log. \frac{x}{a - \sqrt{a^2 - x^2}}$

if we multiply both numerator and denominator by $\sqrt{a + x} - \sqrt{a - x}$ and simplify, $\therefore z = \log x - \log. (a - \sqrt{a^2 - x^2}) \therefore \frac{dz}{dx} = \frac{1}{x} - \frac{x}{(a^2 - x^2)^{\frac{1}{2}}\,(a - \sqrt{a^2 - x^2})} = \frac{a\,(\sqrt{a^2 - x^2} - a)}{x\,(a^2 - x^2)^{\frac{1}{2}}\,(a - \sqrt{a^2 - x^2})} = \frac{- a}{x\,(a^2 - x^2)^{\frac{1}{2}}}$.

Ex. 5. Let $z = a^{x^x}$. Let $y = x^x$, then $\log. y = x \log. x$, and $\frac{dy}{y} = (\log. x + 1)\, dx \therefore dy = x^x\, (\log. x + 1)\, dx$. But $dz = \log. a\, a^{x^x}\, dy \therefore \frac{dz}{dx} = \log. a\, a^{x^x}\, x^x\, (\log. x + 1)$.

Ex. 6. Let $z = a^{b^{(x^2+x)}}$. Let $y = b^{(x^2+x)}$; then
log. $y = (x^2+x)$ log. b ∴ $\frac{dy}{y} = \log. b\,(2x+1)\,dx$ and $dy = \log. b\; b^{(x^2+x)}$ $(2x+1)\,dx$.

But $dz = \log. a\; a^{b^{(x^2+x)}}\, db^{(x^2+x)}$ ∴ $\frac{dz}{dx} = \log. a \log. b\, a^{b^{(x^2+x)}}\, b^{(x^2+x)}\,(2x+1)$.

Examples for Practice.

(1.) Let $z = a^x - \frac{1}{a^x}$ ∴ $\frac{dz}{dx} = \left(a^x + \frac{1}{a^x}\right) \log. a$.

(2.) Let $z = (x-1)\,e^x$ ∴ $\frac{dz}{dx} = xe^x$.

(3.) Let $z = e^{x^x}$ ∴ $\frac{dz}{dx} = e^{x^x}\, x^x\,(1 + \log. x)$.

(4.) Let $z = a^{\log. x}$ ∴ $\frac{dz}{dx} = \frac{\log. a \,.\, a^{\log. x}}{x}$.

(5.) Let $z = \sqrt{\frac{e^x - 1}{e^x + 1}}$ ∴ $\frac{dz}{dx} = \frac{e^x}{(e^x+1)\,(e^{2x}-1)^{\frac{1}{2}}}$.

(6.) Let $z = \log. \frac{(x+2)^2}{x+1}$ ∴ $\frac{dz}{dx} = \frac{x}{x^2+3x+2}$.

(7.) Let $z = \log. (x\sqrt{-1} - \sqrt{1-x^2})$ ∴ $\frac{dz}{dx} = \frac{1}{\sqrt{x^2-1}}$.

(8.) Let $z = \log. \frac{x}{\sqrt{x^2+1}+1}$ ∴ $\frac{dz}{dx} = \frac{1}{x\sqrt{x^2+1}}$.

(9.) Let $z = x^n\, e^{\log. x}$ ∴ $\frac{dz}{dx} = x^{n-1}\,(ne^{\log. x} + x)$.

(10.) Let $z = (\log. x)^n$ ∴ $\frac{dz}{dx} = n\,(\log. x)^{n-1}\,\frac{1}{x}$.

(11.) Let $z = \log. \sqrt{\frac{a^2 - x^2}{a^2 + x^2}}$ $\therefore \frac{dz}{dx} = -\frac{2\,a^2 x}{a^4 - x^4}$.

(12.) Let $z = \log. \frac{e^x - 1}{e^x + 1}$ $\therefore \frac{dz}{dx} = \frac{2\,e^x}{e^{2x} - 1}$.

(13.) Let $z = \log. \frac{\sqrt{1+\sqrt{x}} - 1}{\sqrt{1+\sqrt{x}} + 1}$ $\therefore \frac{dz}{dx} = \frac{1}{2\,x\sqrt{1+\sqrt{x}}}$

(14.) Let $z = \log. \log. x$ $\therefore \frac{dz}{dx} = \frac{1}{x \log. x}$.

SECOND.—TRIGONOMETRICAL OR CIRCULAR FUNCTIONS.

(29.) Let $z = \sin. x$, then $\frac{dz}{dx} = \cos. x$.

For let x become equal to $x + h$, then $z' = \sin. (x + h) = \sin. x + 2 \cos. \left(x + \frac{h}{2}\right) \sin. \frac{h}{2}$ $\therefore \sin. (x + h) - \sin. x = z' - z = 2 \cos. \left(x + \frac{h}{2}\right) \sin. \frac{h}{2}$ $\therefore \frac{z' - z}{h} = \cos. \left(x + \frac{h}{2}\right) \frac{\sin. \frac{1}{2} h}{\frac{1}{2} h}$. Taking the limits of both sides we have $\frac{dz}{dx} = \cos. x$.

(30.) Let $z = \cos. x$, then $\frac{dz}{dx} = - \sin. x$.

For $\cos. x = \sin. \left(\frac{\pi}{2} - x\right)$ $\therefore dz = d \cos. x = \cos. \left(\frac{\pi}{2} - x\right) \times - dx$ $\therefore \frac{dz}{dx} = - \cos. \left(\frac{\pi}{2} - x\right) = - \sin. x$.

(31.) Let $z = \tan. x$, then $\frac{dz}{dx} = \frac{1}{\cos.^2 x} = \sec.^2 x$

For $\tan. x = \frac{\sin. x}{\cos. x} \therefore dz = d \tan. x = \frac{\cos.^2 x + \sin.^2 x}{\cos.^2 x} dx = \frac{1}{\cos.^2 x} dx \therefore \frac{dz}{dx} = \frac{1}{\cos.^2 x} = \sec.^2 x.$

(32.) Let $z = \cot. x$, then $\frac{dz}{dx} = - \frac{1}{\sin.^2 x} = - \operatorname{cosec.}^2 x.$

For $\cot. x = \frac{\cos. x}{\sin. x} \therefore dz = d \cot. x = - \frac{\cos.^2 x + \sin.^2 x}{\sin^2 x} dx = - \frac{1}{\sin.^2 x} dx \therefore \frac{dz}{dx} = - \frac{1}{\sin.^2 x} = - \operatorname{cosec.}^2 x.$

(33.) Let $z = \sec. x$, then $\frac{dz}{dx} = \tan. x \sec. x.$

For $\sec. x = \frac{1}{\cos. x} \therefore dz = d \frac{1}{\cos. x} = \frac{\sin. x\, dx}{\cos.^2 x} = \frac{\sin. x}{\cos. x} \times \frac{1}{\cos. x} dx = \tan. x \sec. x\, dx \therefore \frac{dz}{dx} = \tan. x \sec. x.$

(34.) Let $z = \operatorname{cosec.} x$, then $\frac{dz}{dx} = - \cot. x \operatorname{cosec.} x.$

For $\operatorname{cosec.} x = \frac{1}{\sin. x} \therefore dz = d \frac{1}{\sin. x} = - \frac{\cos. x}{\sin.^2 x} dx = - \frac{\cos. x}{\sin. x} \times \frac{1}{\sin. x} dx = - \cot. x \operatorname{cosec.} x\, dx \therefore \frac{dz}{dx} = - \cot. x \operatorname{cosec.} x.$

(35.) Let $z = \operatorname{versin.} x \therefore \frac{dz}{dx} = \sin. x.$

For $\operatorname{versin.} x = 1 - \cos. x \therefore dz = d\,(1 - \cos. x) = \sin. x\, dx \therefore \frac{dz}{dx} = \sin. x.$

(36.) Let $z = \sin.^{-1} x$ where $\sin.^{-1} x$ represents an arc whose sine is x, then $\frac{dz}{dx} = \frac{1}{\sqrt{1 - x^2}}.$

For since $z = \sin.^{-1}x$ $\therefore$ $x = \sin. z$ $\therefore$ $\cos. z\, dz = dx$.
But $\cos. z = \sqrt{1 - \sin.^2 z} = \sqrt{1 - x^2}$ $\therefore$ $\sqrt{1 - x^2}\, dz = dx$
$\therefore \dfrac{dz}{dx} = \dfrac{1}{\sqrt{1 - x^2}}$.

(37.) Let $z = \cos.^{-1}x$ then $\dfrac{dz}{dx} = -\dfrac{1}{\sqrt{1 - x^2}}$.

For since $z = \cos.^{-1}x$ $\therefore$ $\cos. z = x$ $\therefore$ $-\sin. z\, dz = dx$.
But $\sin. z = \sqrt{1 - \cos.^2 z} = \sqrt{1 - x^2}$ $\therefore$ $-\sqrt{1 - x^2}\, dz = dx$
$\therefore \dfrac{dz}{dx} = -\dfrac{1}{\sqrt{1 - x^2}}$.

(38.) Let $z = \tan.^{-1}x$ then $\dfrac{dz}{dx} = \dfrac{1}{1 + x^2}$.

For since $z = \tan.^{-1}x$, $\tan. z = x$ $\therefore$ $\sec.^2 z\, dz = dx$.

But $\sec.^2 z = 1 + \tan.^2 z = 1 + x^2$ $\therefore$ $(1 + x^2)\, dz = dx$ $\therefore$ $\dfrac{dz}{dx}$
$= \dfrac{1}{1 + x^2}$.

(39.) Let $z = \cot.^{-1}x$, then $\dfrac{dz}{dx} = -\dfrac{1}{1 + x^2}$.

For since $z = \cot.^{-1}x$, $\cot. z = x$ $\therefore$ $-\operatorname{cosec.}^2 z\, dz = dx$.
But $\operatorname{cosec.}^2 z = 1 + \cot.^2 z = 1 + x^2$ $\therefore$ $-(1 + x^2)\, dz = dx$
$\dfrac{dz}{dx} = -\dfrac{1}{1 + x^2}$.

(40.) Let $z = \sec.^{-1}x$, then $\dfrac{dz}{dx} = \dfrac{1}{x\sqrt{x^2 - 1}}$.

For since $z = \sec.^{-1}x$, $\sec. z = x$ $\therefore$ $\tan. z \sec. z dz = dx$.
But $\tan. z = \sqrt{\sec.^2 z - 1} = \sqrt{x^2 - 1}$, and $\sec. z = x$ $\therefore$ $x\sqrt{x^2 - 1}\, dz = dx$
$\dfrac{dz}{dx} = \dfrac{1}{x\sqrt{x^2 - 1}}$.

(41.) Let $z = \operatorname{cosec.}^{-1}x$; then $\dfrac{dz}{dx} = -\dfrac{1}{x\sqrt{x^2 - 1}}$.

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For since $z = \text{cosec.}^{-1}x$ ∴ cosec. $z = x$, and $-$ cot. z cosec. $z\,dz = dx$. But cot. $z = \sqrt{\text{cosec.}^2 z - 1} = \sqrt{x^2 - 1}$, and cosec. $z = x$ ∴ $-x\sqrt{x^2-1}\,dz = dx$ ∴ $\frac{dz}{dx} = -\frac{1}{x\sqrt{x^2-1}}$.

(42.) Let $z = \text{versin.}^{-1}x$, then $\frac{dz}{dx} = \frac{1}{\sqrt{2x - x^2}}$.

For since $z = \text{versin.}^{-1}x$, versin. $z = x$ ∴ sin. $z\,dz = dx$.

But sin. $z = \sqrt{1 - \cos.^2 z} = \sqrt{1 - (1 - \text{versin.}\, z)^2} = \sqrt{1 - (1 - x)^2}$ $= \sqrt{2x - x^2}$ ∴ $\sqrt{2x - x^2}\,dz = dx$ ∴ $\frac{dz}{dx} = \frac{1}{\sqrt{2x - x^2}}$.

(43.) The results in the last 14 articles have been obtained upon the supposition that the radius is unity. We shall exhibit them shortly in the following table, both for the radii 1 and a.

The radius 1.	*The radius* a.
(1.) d sin. x = cos. $x\,dx$.	d sin. $x = \frac{1}{a}$ cos. $x\,dx$.
(2.) d cos. x = sin. $x\,dx$	d cos. $x = -\frac{1}{a}$ sin $x\,dx$
(3.) d tan. x = sec $^2x\,dx$.	d tan. $x = \frac{1}{a^2}$ sec $^2x\,dx$
(4.) d cot. $x = -$ cosec. $^2x\,dx$	d cot $x = -\frac{1}{a^2}$ cosec $^2x\,dx$
(5.) d sec. x = tan. x sec. $x\,dx$.	d sec. $x = \frac{1}{a^2}$ tan. x sec. $x\,dx$.
(6.) d cosec. $x = -$ cot. x cosec. $x\,dx$.	d cosec. $x = -\frac{1}{a^2}$ cot. x cosec. $x\,dx$.
(7.) d versin. x = sin. $x\,dx$.	d versin $x = \frac{1}{a}$ sin $x\,dx$

(8.) $d \sin.^{-1} x = \dfrac{dx}{\sqrt{1-x^2}}$	$d \sin.^{-1} x = \dfrac{a\,dx}{\sqrt{a^2-x^2}}$
(9.) $d \cos.^{-1} x = -\dfrac{dx}{\sqrt{1-x^2}}$	$d \cos.^{-1} x = -\dfrac{a\,dx}{\sqrt{a^2-x^2}}$
(10.) $d \tan.^{-1} x = \dfrac{dx}{1+x^2}$	$d \tan.^{-1} x = \dfrac{a^2\,dx}{a^2+x^2}$
(11.) $d \cot.^{-1} x = -\dfrac{dx}{1+x^2}$	$d \cot.^{-1} x = -\dfrac{a^2\,dx}{a^2+x^2}$
(12.) $d \sec.^{-1} x = \dfrac{dx}{x\sqrt{x^2-1}}$	$d \sec.^{-1} x = \dfrac{a^2\,dx}{x\sqrt{x^2-a^2}}$
(13.) $d \operatorname{cosec.}^{-1} x = -\dfrac{dx}{x\sqrt{x^2-1}}$	$d \operatorname{cosec.}^{-1} x = -\dfrac{a^2\,dx}{x\sqrt{x^2-a^2}}$
(14.) $d \operatorname{versin.}^{-1} x = \dfrac{dx}{\sqrt{2x-x^2}}$	$d \operatorname{versin.}^{-1} x = \dfrac{a\,dx}{\sqrt{2\,ax-x^2}}$

Ex. 1. Let $z = \sin.^3 x$, then $\dfrac{dz}{dx} = 3 \sin.^2 x \cos. x = 3 (\cos. x - \cos.^3 x)$

Ex. 2. Let $z = (2 + \sin.^2 x) \cos. x \therefore \dfrac{dz}{dx} = 2 \sin. x \cos.^2 x - (2 + \sin.^2 x) \sin. x = 2 \sin. x - 2 \sin.^3 x - 2 \sin. x - \sin.^3 x = -3 \sin.^3 x$

Ex. 3. Let $z = y \tan. x^n$, where y is a function of x, then $dz = \tan. x^n\, dy + y \sec.^2 x^n\, n x^{n-1}\, dx \therefore \dfrac{dz}{dx} = \tan. x^n \dfrac{dy}{dx} + ny x^{n-1} \sec.^2 x^n$.

Ex. 4. Let $z = \log. (\sin. x) \therefore dz = \dfrac{d \sin. x}{\sin. x} = \dfrac{\cos. x}{\sin. x} dx \therefore \dfrac{dz}{dx} = \dfrac{\cos. x}{\sin. x} = \cot. x$.

Ex. 5. Let $z = e^{\cos. x} \sin. x$.

$$dz = e^{\cos. x} \sin. x \, d \cos. x + e^{\cos. x} \cos. x \, dx.$$

$$\therefore \frac{dz}{dx} = - e^{\cos. x} \sin. {}^2 x + e^{\cos. x} \cos. x = e^{\cos. x} (\cos. x - \sin. {}^2 x)$$

$$= e^{\cos. x} (\cos. {}^2 x + \cos. x - 1).$$

Ex. 6. Let $z = \log. \sqrt{\frac{1 + \cos. x}{1 - \cos. x}} = \frac{1}{2} \log. (1 + \cos. x) - \frac{1}{2} \log (1 - \cos. x)$ $\quad \therefore \frac{dz}{dx} = - \frac{\sin. x}{2 (1 + \cos. x)} - \frac{\sin. x}{2 (1 - \cos. x)} =$

$$- \frac{\sin. x + \sin. x \cos. x + \sin. x - \sin. x \cos. x}{2 (1 - \cos. {}^2 x)} = - \frac{\sin. x}{1 - \cos. {}^2 x} =$$

$$- \frac{\sin. x}{\sin. {}^2 x} = - \frac{1}{\sin. x}.$$

Ex. 7. Let $z = \sin.^{-1} \frac{x}{\sqrt{1 + x^2}}$

$$\therefore \sin. z = \frac{x}{(1 + x^2)^{\frac{1}{2}}} \quad \therefore \cos. z \frac{dz}{dx} = \frac{(1 + x^2)^{\frac{1}{2}} - \frac{1}{2} (1 + x^2)^{-\frac{1}{2}} 2 x}{1 + x^2}$$

$$= \frac{1 + x^2 - x^2}{(1 + x^2)^{\frac{3}{2}}} = \frac{1}{(1 + x^2)^{\frac{3}{2}}}. \quad \text{But } \cos. z = \sqrt{1 - \sin. {}^2 z}$$

$$= \sqrt{1 - \frac{x^2}{1 + x^2}} = \frac{1}{(1 + x^2)^{\frac{1}{2}}} \quad \therefore \frac{dz}{dx} = \frac{1}{1 + x^2}.$$

Ex. 8. Let $z = \tan^{-1} \sqrt{\frac{a + bx}{b - a}} \quad \therefore \tan. z = \frac{(a + bx)^{\frac{1}{2}}}{(b - a)^{\frac{1}{2}}}$

$$\therefore \sec. {}^2 z \frac{dz}{dx} = \frac{b}{2 (b - a)^{\frac{1}{2}} (a + bx)^{\frac{1}{2}}}. \quad \text{But } \sec. {}^2 z = 1 + \tan. {}^2 z$$

$$= 1 + \frac{a + bx}{(b - a)} = \frac{b (1 + x)}{b - a} \quad \therefore \frac{dz}{dx} = \frac{b - a}{b (1 + x)} \times \frac{b}{2 (b - a)^{\frac{1}{2}} (a + bx)^{\frac{1}{2}}}$$

$$= \frac{(b - a)^{\frac{1}{2}}}{2 (1 + x) (a + bx)^{\frac{1}{2}}}.$$

EXAMPLES FOR PRACTICE.

(1.) Let $z = \dfrac{1 - \cos. x}{\cos.^2 x}$ $\qquad\therefore \dfrac{dz}{dx} = \dfrac{\sin. x\,(2 - \cos. x)}{\cos.^3 x}$.

(2.) Let $z = \sec.^n x$ $\qquad\therefore \dfrac{dz}{dx} = \dfrac{n \sin. x}{\cos.^{n+1} x}$.

(3.) Let $z = \log. \sec.^2 x$ $\qquad\therefore \dfrac{dz}{dx} = 2 \tan. x$.

(4.) Let $z = \sin. \log. x$ $\qquad\therefore \dfrac{dz}{dx} = \dfrac{\cos. \log. x}{x}$.

(5.) Let $z = \log. \sqrt{\dfrac{1 + \sin.^2 x}{1 - \sin.^2 x}}$ $\qquad\therefore \dfrac{dz}{dx} = \dfrac{\sin. 2x}{1 - \sin.^4 x}$.

(6.) Let $z = \log. \left(\dfrac{1 + \sqrt{-1} \tan. x}{1 - \sqrt{-1} \tan. x}\right)$ $\qquad\therefore \dfrac{dz}{dx} = 2\sqrt{-1}$.

(7.) Let $z = \log. (\cos. x + \sqrt{-1} \sin. x)$ $\qquad\therefore \dfrac{dz}{dx} = \sqrt{-1}$.

(8.) Let $z = \tan.^{-1} \dfrac{2x}{1 - x^2}$ $\qquad\therefore \dfrac{dz}{dx} = \dfrac{2}{1 + x^2}$.

(9.) Let $z = \sin.^{-1} \dfrac{x - 1}{2^{\frac{1}{2}}}$ $\qquad\therefore \dfrac{dz}{dx} = \dfrac{1}{(1 + 2x - x^2)^{\frac{1}{2}}}$.

(10.) Let $z = \cot.^{-1} \sqrt{\dfrac{1 - x}{x}}$ $\qquad\therefore \dfrac{dz}{dx} = \dfrac{1}{2\sqrt{x - x^2}}$.

(11.) Let $z = \cos.^{-1} \left(\dfrac{b + a \cos. x}{a + b \cos. x}\right)$ $\qquad\therefore \dfrac{dz}{dx} = \dfrac{(a^2 - b^2)^{\frac{1}{2}}}{a + b \cos. x}$.

(12.) Let $z = \tan.^{-1} \left(\left(\dfrac{a - b}{a + b}\right)^{\frac{1}{2}} \tan. x\right)$ $\qquad\therefore \dfrac{dz}{dx} = \dfrac{(a^2 - b^2)^{\frac{1}{2}}}{a + b \cos. 2x}$.

CHAPTER III.

SUCCESSIVE DIFFERENTIATION AND ELIMINATION OF CONSTANTS AND FUNCTIONS.

(44.) Let $z = f(x)$, then $\frac{dz}{dx} = f'(x)$ = a new function of x, $\frac{d^2z}{dx^2} = f''(x)$, $\frac{d^3z}{dx^3} = f'''(x)$, &c. = &c.

Ex. Let $z = 5\,ax^4$, then $\frac{dz}{dx} = 20\,ax^3 = f'(x)$, $\frac{d^2z}{dx^2} = 60\,ax^2 = f''(x)$, $\frac{d^3z}{dx^3} = 120\,ax = f'''(x)$, $\frac{d^4z}{dx^4} = 120\,a$, which terminates the differentiation, as $120\,a$ does not contain x.

(45.) To find the successive differential coefficients of x^n.

Let $z = x^n$, then $\frac{dz}{dx} = n\,x^{n-1}$, $\frac{d^2z}{dx^2} = n\overline{n-1}\,x^{n-2}$

$$\frac{d^3z}{dx^3} = n\overline{n-1}\,\overline{n-2}\,x^{n-3},\ \frac{d^rz}{dx^r} = n\overline{n-1}\ \ldots\ \overline{n-r+1}\,x^{n-r},$$

$$\frac{d^nz}{dx^n} = n\,\overline{n-1}\ \ldots\ 3\,.\,2\,.\,1.$$

(46.) To find the successive differential coefficients of a^x.

Let $z = a^x$, then $\frac{dz}{dx} = \log.a\,a^x$, $\frac{d^2z}{dx^2} = \log.^2a\,a^x = A^2a^x$, $\frac{d^nz}{dx^n} = A^n a^x$.

(47.) To find the differential coefficients of sin. x and cos. x.

$$\frac{d\sin.x}{dx} = \cos x \qquad \frac{d\cos.x}{dx} = -\sin.x,$$

$$\frac{d^2\sin.x}{dx^2} = -\sin.x \qquad \frac{d^2\cos.x}{dx^2} = -\cos.x.$$

$$\frac{d^3\sin.x}{dx^3} = -\cos.x \qquad \frac{d^3\cos.x}{dx^3} = \sin.x$$

$$\frac{d^4\sin x}{dx^4} = \sin x \qquad \frac{d^4\cos x}{dx^4} = \cos.x.$$

(48.) To find the successive differential coefficients of $\sin.^{-1}x$.

Let $z = \sin.^{-1}x$, then $\dfrac{dz}{dx} = \dfrac{1}{(1-x^2)^{\frac{1}{2}}}$, $\dfrac{d^2z}{dx^2} = \dfrac{x}{(1-x^2)^{\frac{3}{2}}}$

$\dfrac{d^3z}{dx^3} = \dfrac{1+2x^2}{(1-x^2)^{\frac{5}{2}}}$, &c. = &c.

(49.) To find the successive differential coefficients of uv, where u and v are functions of x.

Let $z = uv$, then

$$\frac{dz}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d^2z}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + \frac{d^2u}{dx^2}v$$

$$\frac{d^3z}{dx^3} = u\frac{d^3v}{dx^3} + 3\frac{du}{dx}\frac{d^2v}{dx^2} + 3\frac{d^2u}{dx^2}\frac{dv}{dx} + \frac{d^3u}{dx^3}v$$

&c. = &c.

Here the law of the exponents and coefficients is the same as that of $(u+v)^n$ $\therefore$ $\dfrac{d^nz}{dx^n} = u\dfrac{d^nv}{dx^n} + n\dfrac{du}{dx}\dfrac{d^{n-1}v}{dx^{n-1}} + \dfrac{n\,\overline{n-1}}{1\,.\,2}\dfrac{d^2u}{dx^2}\dfrac{d^{n-2}v}{dx^{n-2}}$

$+\dfrac{n\,\overline{n-1}\,\overline{n-2}}{1\,.\,2\,.\,3}\dfrac{d^3u}{dx^3}\dfrac{d^{n-3}v}{dx^{n-3}}$ + &c., which is Leibnitz's Theorem.

To demonstrate this theorem, since we have

$$\frac{dz}{dx} = u\frac{dv}{dx} + \frac{du}{dx}v.$$

If we separate the symbols of operation from those of quantity, and make $\frac{d}{dx}$ and $\frac{d'}{dx}$ represent the symbols of differentiation of v and u respectively, we have

$$\frac{dz}{dx} = \left(\frac{d}{dx} + \frac{d'}{dx}\right)uv.$$

Let n now represent the index of operation on both sides, and we have

$$\frac{d^n z}{dx^n} = \left(\frac{d}{dx} + \frac{d'}{dx}\right)^n uv = u\frac{d^n v}{dx^n} + n\frac{du}{dx}\frac{d^{n-1}v}{dx^{n-1}}$$

$$+ \frac{n\,\overline{n-1}}{1\,.\,2}\frac{d^2u}{dx^2}\frac{d^{n-2}v}{dx^{n-2}} + \frac{n\,\overline{n-1}\,\overline{n-2}}{1\,.\,2\,.\,3}\frac{d^3u}{dx^3}\frac{d^{n-3}v}{dx^{n-3}} + \&c.$$

This demonstration applies whether n be whole or fractional, positive or negative; while that given by Leibnitz establishes the truth of the theorem for integer indices only.

Vide Leibnitz Commer. Epis. Vol. I. page 46.

THE ELIMINATION OF CONSTANTS AND FUNCTIONS BY DIFFERENTIATION.

(50.) Let $z = ax^2 + bx$ (1)

$$\frac{dz}{dx} = 2ax + b \qquad (2)$$

$$\frac{d^2z}{dx^2} = 2a \qquad (3).$$

(1) is called the *primitive* equation; (2) a *derived* equation of the *first* order; (3) a *derived* equation of the *second* order, and so on.

Vide Lagrange Calcul. des Fonctions, page 151.

(51.) As there are two constants in the primitive equation, and we have now three equations, we may obtain an equation of the second order, in which no constant shall appear.

Thus $a = \frac{d^2z}{2\,dx^2}$, $b = \frac{dz}{dx} - \frac{d^2z}{dx^2}\,x$,

$$\therefore \frac{d^2z}{dx^2} - \frac{dz}{dx}\,\frac{2}{x} + \frac{2\,z}{x^2} = 0.$$

From this it appears that the first derived equation enables us to eliminate one constant; the second, an additional constant; and so on. It hence appears, that whatever be the number of constants in any one equation, they may be eliminated by Differentiation. Fractional quantities and Transcendental functions may be eliminated in a similar manner.

Example (1.) Eliminate m and a from the equation

$$z^2 = m\,(a^2 - x^2)$$

$$2\,z\,\frac{dz}{dx} = -\,2\,mx$$

$$\frac{z}{x}\,\frac{dz}{dx} = -\,m$$

$$\therefore \frac{z}{x}\,\frac{d^2z}{dx^2} + \frac{1}{x}\left(\frac{dz}{dx}\right)^2 - \frac{z}{x^2}\,\frac{dz}{dx} = 0$$

$$xz\,\frac{d^2z}{dx^2} + x\left(\frac{dz}{dx}\right)^2 - z\,\frac{dz}{dx} = 0.$$

Ex. (2.) Eliminate c from the equation $x - y = c\,e^{-\frac{x}{x-y}}$

$$\log.\,(x - y) = -\,\frac{x}{x - y} + \log.\,c$$

$$\frac{1 - \frac{dy}{dx}}{x - y} = \frac{y - x\,\frac{dy}{dx}}{(x - y)^2}$$

$$\therefore x - 2\,y + y\,\frac{dy}{dx} = 0.$$

Ex. (3.) Eliminate the constants and functions from

$$y = a\,\sin.\,x - b\,\cos.\,x.$$

$$\frac{dy}{dx} = a \cos. x + b \sin. x.$$

$$\frac{d^2y}{dx^2} = -a \sin. x + b \cos. x$$

$$\therefore \frac{d^2y}{dx^2} + y = 0.$$

Ex. (4.) Eliminate the exponential and circular functions from

$$y = a\, e^{mx} \sin. nx$$

$$\log. y = \log. a + mx + \log. \sin. nx.$$

Differentiate $\frac{1}{y}\frac{dy}{dx} = m + n \cot. nx$

$$\frac{1}{y}\frac{d^2y}{dx^2} - \frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 = - n^2 \operatorname{cosec.}^2 nx$$

$$= - n^2 - n^2 \cot.^2 nx.$$

But $n^2 \cot.^2 nx = \frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 - \frac{2\,m}{y}\frac{dy}{dx} + m^2$

$$\therefore \frac{d^2y}{dx^2} - 2\,m\frac{dy}{dx} + (m^2 + n^2)\, y = 0.$$

CHAPTER IV.

DEVELOPMENT OF FUNCTIONS OF ONE VARIABLE.

MACLAURIN'S THEOREM.

(52.) Let $z = f(x)$ and let $f(x) = A + Bx + Cx^2 + Dx^3 + \&c.$,

then $\frac{dz}{dx} = B + 2Cx + 3Dx^2 + \ldots$, $\frac{d^2z}{dx^2} = 2C + 3.2.Dx + \ldots$,

$\frac{d^3z}{dx^3} = 3.2.D + \ldots$

Let the values assumed by z, $\frac{dz}{dx}$, $\frac{d^2z}{dx^2}$, $\frac{d^3z}{dx^3}$, &c. when $x = 0$, be represented by (z), $\left(\frac{dz}{dx}\right)$, $\left(\frac{d^2z}{dx^2}\right)$, $\left(\frac{d^3z}{dx^3}\right)$, &c.

Then $(z) = A$, $\left(\frac{dz}{dx}\right) = B$, $\frac{1}{2}\left(\frac{d^2z}{dx^2}\right) = C$, $\frac{1}{1.2.3}\left(\frac{d^3z}{dx^3}\right) = D$

$\therefore z = (z) + \left(\frac{dz}{dx}\right)\frac{x}{1} + \left(\frac{d^2z}{dx^2}\right)\frac{x^2}{1.2} + \left(\frac{d^3z}{dx^3}\right)\frac{x^3}{1.2.3} + \&c.$ which is Maclaurin's Theorem.

Ex. (1.) Let $z = \sqrt{a+x}$. Then

$$\frac{dz}{dx} = \frac{1}{2(a+x)^{\frac{1}{2}}}$$

$$\frac{d^2z}{dx^2} = -\frac{1}{4(a+x)^{\frac{3}{2}}}$$

$$\frac{d^3z}{dx^3} = \frac{3}{8(a+x)^{\frac{5}{2}}}$$

&c. = &c.

Making $x = 0$ in the values of z, $\frac{dz}{dx}$, $\frac{d^2z}{dx^2}$, $\frac{d^3z}{dx^3}$, &c., we obtain

$$(z) = a^{\frac{1}{2}}, \left(\frac{dz}{dx}\right) = \frac{1}{2a^{\frac{1}{2}}}, \left(\frac{d^2z}{dx^2}\right) = -\frac{1}{4a^{\frac{3}{2}}}, \left(\frac{d^3z}{dx^3}\right) = \frac{3}{8a^{\frac{5}{2}}}$$

$$\therefore z = \sqrt{a+x} = a^{\frac{1}{2}} + \frac{1}{2}\frac{x}{a^{\frac{1}{2}}} - \frac{1}{8}\frac{x^2}{a^{\frac{3}{2}}} + \frac{1}{16}\frac{x^3}{a^{\frac{5}{2}}} - \text{\&c.}$$

$$= a^{\frac{1}{2}}\left(1 + \frac{1}{2}\frac{x}{a} - \frac{1}{8}\frac{x^2}{a^2} + \frac{1}{16}\frac{x^3}{a^3} - \text{\&c.}\right)$$

Ex. (2.) Let $z = (a+x)^n$ then $\frac{dz}{dx} = n(a+x)^{n-1}$

$$\frac{d^2z}{dx^2} = n\,\overline{n-1}\,(a+x)^{n-2},\ \frac{d^3z}{dx^3} = n\,\overline{n-1}\,\overline{n-2}\,(a+x)^{n-3}, \text{\&c.} = \text{\&c.}$$

Let $x = 0$ in the values of z, $\frac{dz}{dx}$, $\frac{d^2z}{dx^2}$, $\frac{d^3z}{dx^3}$, &c.

$$\therefore (z) = a^n, \left(\frac{dz}{dx}\right) = na^{n-1}, \left(\frac{d^2z}{dx^2}\right) = n\,\overline{n-1}\,a^{n-2}, \left(\frac{d^3z}{dx^3}\right) =$$

$$n\,\overline{n-1}\,\overline{n-2}\,a^{n-3}\ \therefore z = (a+x)^n = a^n + na^{n-1}x + \frac{n\,\overline{n-1}}{1\,.\,2}a^{n-2}x^2 +$$

$$\frac{n\,\overline{n-1}\,\overline{n-2}}{1\,.\,2\,.\,3}a^{n-3}x^3 + \text{\&c.}$$ which is the Binomial Theorem.

(53.) To expand sin. x and cos. x in terms of x.

$$z = \sin. x \qquad\qquad z = \cos. x$$

$$\frac{dz}{dx} = \cos. x \qquad\qquad \frac{dz}{dx} = -\sin. x$$

$$\frac{d^2z}{dx^2} = -\sin. x \qquad\qquad \frac{d^2z}{dx^2} = -\cos. x$$

$$\frac{d^3z}{dx^3} = -\cos. x \qquad\qquad \frac{d^3z}{dx^3} = \sin. x$$

$$\frac{d^4z}{dx^4} = \sin. x \qquad\qquad \frac{d^4z}{dx^4} = \cos. x$$

$$\frac{d^5z}{dx^5} = \cos. x \qquad\qquad \frac{d^5z}{dx^5} = -\sin. x$$

$$\frac{d^6z}{dx^6} = -\sin. x \qquad\qquad \frac{d^6z}{dx^6} = -\cos. x$$

&c. = &c. &c. = &c.

Let $x = 0$ in the values of $z, \frac{dz}{dx}, \frac{d^2z}{dx^2}, \frac{d^3z}{dx^3}$, &c., then

$$(z) = 0 \qquad\qquad (z) = 1$$

$$\left(\frac{dz}{dx}\right) = 1 \qquad\qquad \left(\frac{dz}{dx}\right) = 0$$

$$\left(\frac{d^2z}{dx^2}\right) = 0 \qquad\qquad \left(\frac{d^2z}{dx^2}\right) = -1$$

$$\left(\frac{d^3z}{dx^3}\right) = -1 \qquad\qquad \left(\frac{d^3z}{dx^3}\right) = 0$$

$$\left(\frac{d^4z}{dx^4}\right) = 0 \qquad\qquad \left(\frac{d^4z}{dx}\right) = 1$$

$$\left(\frac{d^5z}{dx^5}\right) = 1 \qquad\qquad \left(\frac{d^5z}{dx^5}\right) = 0$$

$$\left(\frac{d^6z}{dx^6}\right) = 0 \qquad\qquad \left(\frac{d^6z}{dx^6}\right) = -1$$

$$\therefore \sin. x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \&c.$$

$$\text{and } \cos. x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \&c.$$

$$\text{But } e^{x\sqrt{-1}} = 1 + x\sqrt{-1} - \frac{x^2}{1.2} - \frac{x^3\sqrt{-1}}{1.2.3} + \&c.$$

$$\text{and } e^{-x\sqrt{-1}} = 1 - x\sqrt{-1} - \frac{x^2}{1.2} + \frac{x^3\sqrt{-1}}{1.2.3} + \&c.$$

$$\therefore \tfrac{1}{2}\left(e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}\right) = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \&c. = \cos. x,$$

$$\text{and } \frac{1}{2\sqrt{-1}}\left(e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}\right) = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \&c. = \sin. x.$$

$$\text{But } \tan. x = \frac{\sin. x}{\cos. x} = \frac{1}{\sqrt{-1}}\,\frac{e^{2x\sqrt{-1}} - 1}{e^{2x\sqrt{-1}} + 1}$$

Also $\cos. x \pm \sqrt{-1}\sin. x = e^{\pm x\sqrt{-1}}$. Let x become $= mx$, then $\cos. mx \pm \sqrt{-1}\sin. mx = e^{\pm mx\sqrt{-1}} = (e^{\pm x\sqrt{-1}})^m = (\cos. x \pm \sqrt{-1}\sin. x)^m$, which is *De Moivre's Theorem.*

(54.) Expand $\tan. x$ in ascending powers of x.

This might be done by Maclaurin's Theorem, but we will adopt the following process :—Since $(\tan. x)_{x=0} = 0$, x will appear in each term of the expansion.

* Let $\tan. x = ax + bx^3 + cx^5 + dx^7 + \&c. =$

$$\frac{x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \&c.}{1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \&c.}$$

Multiplying by the denominator, and equating the coefficients, we have

$$a = 1,\ b = \frac{2}{1.2.3},\ c = \frac{16}{1.2.3.4.5},\ \&c. = \&c. \quad \therefore$$

$$\tan. x = x + \frac{2\,x^3}{1.2.3} + \frac{16x^5}{1.2.3.4.5} + \&c.$$

(55.) Expand $\sin.^{-1} x$ in ascending powers of x.

Since $(\sin.^{-1} x)_{x=0} = 0 \ \therefore x$ appears in all the terms.

Let $\sin.^{-1} x = ax + bx^2 + cx^3 + ex^4 + fx^5 + \&c.$

$$\frac{1}{\sqrt{1 - x^2}} = a + 2bx + 3cx^2 + 4ex^3 + 5fx^4 + \&c.$$

But $\dfrac{1}{\sqrt{1 - x^2}} = 1 + \frac{1}{2}x^2 + \dfrac{1.3}{2.4}x^4 + \dfrac{1.3.5\,x^6}{2.4.6} + \&c.$

$$\therefore a = 1,\ b = 0,\ c = \frac{1}{2.3},\ e = 0,\ f = \frac{1.3}{2.4.5},\ \&c.$$

$$\therefore \sin.^{-1} x = x + \frac{1}{2.3}x^3 + \frac{1.3}{2.4.5}x^5 + \&c.$$

(56.) Expand $\tan.^{-1} x$ in ascending powers of x.

Since $(\tan.^{-1} x)_{x=0} = 0 \ \therefore x$ appears in all the terms, and it may be proved, as in (55.), that the expansion contains only odd powers of x.

---

* The expansion for $\tan. x$ contains only the odd powers of x, since $\tan. - x = - \tan. x$.

Let $\tan.^{-1} x = ax + bx^3 + cx^5 + dx^7 + ex^9$ &c.

$$\therefore \frac{1}{1+x^2} = a + 3\,bx^2 + 5\,cx^4 + 7\,dx^6 + 9\,ex^8 + \text{\&c.}$$

$$\text{But } \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \text{\&c.}$$

$$\therefore a = 1,\ b = -\frac{1}{3},\ c = \frac{1}{5},\ d = -\frac{1}{7},\ e = \frac{1}{9},\ \text{\&c.}$$

$$\therefore \tan.^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \frac{1}{11}x^{11} + \text{\&c.}$$

Let $x = 1$, then $\tan.^{-1} 1 = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{\&c.}$

This series would enable us to obtain an approximation to the circumference of a circle whose radius is 1. But as it converges very slowly, it is not well adapted for that purpose. It may be rendered more suitable as follows:—

Since $\tan.^{-1} t_1 + \tan.^{-1} t_2 = \tan.^{-1} \frac{t_1 + t_2}{1 - t_1 t_2}$. Let $t_1 = \frac{1}{2}$, $t_2 = \frac{1}{3}$

and we have $\tan.^{-1} 1 = \tan.^{-1} \frac{1}{2} + \tan.^{-1} \frac{1}{3}$. But $\tan.^{-1} 1 = \frac{\pi}{4}$

$$\therefore \frac{\pi}{4} = \left\{ \begin{array}{l} \frac{1}{2} - \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} - \frac{1}{7 \cdot 2^7} + \text{\&c.} \\ + \frac{1}{3} - \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} - \frac{1}{7 \cdot 3^7} + \text{\&c.,} \end{array} \right.$$ which is Euler's series.

Again, since $2a = \tan.^{-1} \frac{2 \tan.\, a}{1 - \tan.^2 a}$ let $\tan.\, a = \frac{1}{5}$ then $2a = \tan.^{-1} \frac{5}{12}$

$\therefore 2a < \frac{\pi}{4}$ because $\frac{\pi}{4} = \tan.^{-1} 1.$

Again, $4a = \tan.^{-1}\dfrac{2\tan.2a}{1-\tan.^2 2a} = \tan.^{-1}\dfrac{120}{119} \therefore 4a > \dfrac{\pi}{4}$.

Let $A = 4a$, then $\tan.(A - 45) = \dfrac{\tan.A - 1}{\tan.A + 1} \therefore A - 45 =$

$\tan.^{-1}\dfrac{\tan.A - 1}{\tan.A+1} = \dfrac{1}{239} \therefore \tan.^{-1} 1 = 4\tan.^{-1}\dfrac{1}{5} - \tan.^{-1}\dfrac{1}{239}$, and

$$\therefore \frac{\pi}{4} = \left\{ \begin{array}{l} 4\left(\frac{1}{5} - \frac{1}{3\,.\,5^3} + \frac{1}{5\,.\,5^5} - \frac{1}{7\,.\,5^7} + \&c.\right) \\ -\left(\frac{1}{239} - \frac{1}{3\,(239)^3} + \frac{1}{5\,(239)^5} - \frac{1}{7\,(239)^7} + \&c.\right) \end{array} \right.$$

which is Machin's series. It is still more convergent than that of Euler. If we take 8 terms in the first row, and 3 in the second, we will find the circumference of a circle to the diameter 1, or the semi-circumference to the radius $1 = 3.14159265358979З = \pi$.

(57.) Let $y = 1 + xe^y$, it is required to expand y in terms of x, by Maclaurin's Theorem.

When $x = 0$, $y = 1$,

$$\frac{dy}{dx} = e^y + xe^y\frac{dy}{dx} \therefore \left(\frac{dy}{dx}\right) = e$$

$$\frac{d^2y}{dx^2} = \frac{e^y\dfrac{dy}{dx}(1 - xe^y) + e^y\left(e^y + xe^y\dfrac{dy}{dx}\right)}{(1-xe^y)^2} \therefore \left(\frac{d^2y}{dx^2}\right) = 2e^2.$$

In a similar manner it appears that $\left(\dfrac{d^3y}{dx^3}\right) = 9e^3$ and $\left(\dfrac{d^4y}{dx^4}\right) = 64e^4$.

$$\therefore y = 1 + ex + \frac{2e^2x^2}{1\,.\,2} + \frac{9e^3x^3}{1\,.\,2\,.\,3} + \frac{64e^4x^4}{1\,.\,2\,.\,3\,.\,4} + \ldots$$

(58.) Expand e^{ax} cos. nx by Maclaurin's Theorem.

$$\log. z = ax + \log. \cos. nx \therefore \frac{dz}{dx} = e^{ax} (a \cos. nx - n \sin. nx)$$

Let $\tan. \varphi = \frac{n}{a}$, then $a = (a^2 + n^2)^{\frac{1}{2}} \cos. \varphi$ and $n = (a^2 + n^2)^{\frac{1}{2}} \sin. \varphi$,

$$\therefore \frac{dz}{dx} = e^{ax}(a^2 + n^2)^{\frac{1}{2}} (\cos. \varphi \cos. nx - \sin. \varphi \sin. nx) = e^{ax} (a^2 + n^2)^{\frac{1}{2}}$$
$$\cos. (nx + \varphi)$$

$\therefore \left(\frac{dz}{dx}\right) = (a^2 + n^2)^{\frac{1}{2}} \cos. \varphi$. In a similar manner it appears that

$$\left(\frac{d^2z}{dx^2}\right) = (a^2 + n^2) \cos. 2\varphi, \left(\frac{d^3z}{dx^3}\right) = (a^2 + n^2)^{\frac{3}{2}} \cos. 3\varphi, \&c. = \&c.$$

$$\therefore e^{ax} \cos. nx = 1 + (a^2 + n^2)^{\frac{1}{2}} \cos. \varphi \frac{x}{1} + (a^2 + n^2) \cos. 2\varphi \frac{x^2}{1 . 2} +$$
$$(a^2 + n^2)^{\frac{3}{2}} \cos. 3\varphi . \frac{x^3}{1 . 2 . 3} + \ldots$$

Taylor's Theorem.

(59.) Let $z = f(x)$ and $z' = f(x + h)$ then

$$z' = z + \frac{dz}{dx}\frac{h}{1} + \frac{d^2z}{dx^2}\frac{h^2}{1 . 2} + \frac{d^3z}{dx^3}\frac{h^3}{1 . 2 . 3} + \frac{d^4z}{dx^4}\frac{h^4}{1 . 2 . 3 . 4} + \ldots$$

Before proceeding to the demonstration of this theorem, we must prove that if $z = f(x)$, and x be changed into $x + h$, we will have the same differential coefficient, whether we regard x variable and h constant, or h variable and x constant.

For let $z = f(x)$ and $z' = f(x + h)$, then $\frac{dz'}{dx} = \varphi (x + h)$ if x be

variable and h constant, and $\frac{dz'}{dh} = \varphi(x + h)$ if h be variable and x constant.

$$\therefore \frac{dz'}{dx} = \frac{dz'}{dh}.$$

Now let $z' = f(x + h) = z + Ah + Bh^2 + Ch^3 + \&c.$ (188) where A, B, C, &c. are unknown functions of x, which we wish to determine. For this purpose let us differentiate with respect to h, and we obtain $\frac{dz'}{dh} = A + 2\,Bh + 3\,Ch^2 + \&c.$

Let us differentiate with respect to x, and $\frac{dz'}{dx} = \frac{dz}{dx} + \frac{dA}{dx}h + \frac{dB}{dx}h^2 + \&c.$

But $\frac{dz'}{dh} = \frac{dz'}{dx}$ $\therefore A + 2Bh + 3Ch^2 + \&c. = \frac{dz}{dx} + \frac{dA}{dx}h + \frac{dB}{dx}h^2 + \&c.$

$$\therefore A = \frac{dz}{dx},\ B = \frac{dA}{2dx} = \frac{d^2z}{dx^2}\frac{1}{1\,.\,2},\ C = \frac{dB}{3dx} = \frac{d^3z}{dx^3}\frac{1}{1\,.\,2\,.\,3}$$

$$\therefore z' = f(x + h) = z + \frac{dz}{dx}h + \frac{d^2z}{dx^2}\frac{h^2}{1\,.\,2} + \frac{d^3z}{dx^3}\frac{h^3}{1\,.\,2\,.\,3} + \&c.,$$

which is Taylor's Theorem.

This theorem may be written as follows:—

$$f(x + h) = f(x) + \frac{d}{dx}f(x)h + \frac{d^2}{dx^2}f(x)\frac{h^2}{1\,.\,2} + \frac{d^3}{dx^3}f(x)\frac{h^3}{1\,.\,2\,.\,3}$$

+ &c., and separating the symbols of operation from that of quantity.

$$f(x + h) = (1 + \frac{d}{dx}\frac{h}{1} + \frac{d^2}{dx^2}\frac{h^2}{1\,.\,2} + \frac{d^3}{dx^3}\frac{h^3}{1\,.\,2\,.\,3} + \&c.)\ f(x)$$

$\therefore f(x+h) = \varepsilon^{h\frac{d}{dx}} f(x)$ by the exponential theorem $= E^h f(x)$ if $E = \varepsilon^{\frac{d}{dx}}$.

Lagrange writes this theorem in the following manner:

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{1.2} + f'''(x)\frac{h^3}{1.2.3} + f''''(x)\frac{h^4}{1.2.3.4} + \&c.$$

Vide Theorie Des Fonctions Analytique, page 18.

Ex. (1.) Let $z' = (x+h)^{\frac{2}{3}}$ $\therefore z = x^{\frac{2}{3}}$, $\frac{dz}{dx} = \frac{2}{3x^{\frac{1}{3}}}$, $\frac{d^2z}{dx^2} = -\frac{2}{9x^{\frac{4}{3}}}$, $\frac{d^3z}{dx^3} = \frac{8}{27x^{\frac{7}{3}}}$, &c. = &c.

$$\therefore z' = (x+h)^{\frac{2}{3}} = x^{\frac{2}{3}} + \frac{2}{3}\frac{h}{x^{\frac{1}{3}}} - \frac{2}{9}\frac{h^2}{x^{\frac{4}{3}}.1.2} + \frac{8\,h^3}{27\,x^{\frac{7}{3}}.1.2.3} - \&c.$$

$$= x^{\frac{2}{3}}\left(1 + \frac{2}{3}\frac{h}{x} - \frac{1}{9}\frac{h^2}{x^2} + \frac{4}{81}\frac{h^3}{x^3}\right) - \&c.,$$ which result coincides with that found by the Binomial Theorem.

Ex. 2. Let $z' = \sin.(x+h)$ $\therefore z = \sin.x$, $\frac{dz}{dx} = \cos.x$

$\frac{d^2z}{dx^2} = -\sin.x$, $\frac{d^3z}{dx^3} = -\cos.x$, $\frac{d^4z}{dx^4} = \sin.x$, $\frac{d^5z}{dx^5} = \cos.x$.

Hence $z' = \sin.(x+h) = \sin.x + \cos.x\,h - \sin x\frac{h^2}{1.2} - \cos.x\frac{h^3}{1.2.3} + \sin.x\frac{h^4}{1.2.3.4} + \cos.x\frac{h^5}{1.2.3.4.5} - \&c. = \sin.x\left(1 - \frac{h^2}{1.2} + \frac{h^4}{1.2.3.4} - \&c.\right)$

$+ \cos. x \left(h - \frac{h^3}{1 \cdot 2 \cdot 3} + \frac{h^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \text{\&c.}\right)$ But $\sin. (x + h) =$

$\sin. x \cos. h + \cos. x \sin. h \therefore \sin. h = h - \frac{h^3}{1 \cdot 2 \cdot 3} + \frac{h^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$

$- \text{\&c.}$, and $\cos. h = 1 - \frac{h^2}{1 \cdot 2} + \frac{h^4}{1 \cdot 2 \cdot 3 \cdot 4} - \text{\&c.}$

(60.) Let $z' = \log. (x + h) \therefore z = \log. x, \frac{dz}{dx} = \frac{1}{x}, \frac{d^2z}{dx^2} = -\frac{1}{x^2}$

$\frac{d^3z}{dx^3} = \frac{2}{x^3}, \frac{d^4z}{dx^4} = -\frac{2 \cdot 3}{x^4}$, &c. = &c.

$\therefore z' = \log. (x + h) = \log. x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \frac{h^4}{4x^4} + \text{\&c.}$

a series which converges very fast if h be small compared with x. It may be better adapted for calculation by the following process:—Let $x = 1$, then $\log. (x + h) = \log. (1 + h) =$

$$0 + h - \frac{h^2}{2} + \frac{h^3}{3} - \frac{h^4}{4} + \text{\&c.}$$

and $\log. (1 - h) = 0 - h - \frac{h^2}{2} - \frac{h^3}{3} - \frac{h^4}{4} - \text{\&c.}$

$\therefore \log. \frac{1 + h}{1 - h} = 2 \left(h + \frac{h^3}{3} + \frac{h^5}{5} + \frac{h^7}{7} + \text{\&c.}\right)$

Let $h = \frac{1}{2m + 1}$ then $1 + h = \frac{2m + 2}{2m + 1}$ and $1 - h = \frac{2m}{2m + 1}$.

$\therefore \frac{1 + h}{1 - h} = \frac{m + 1}{m}$ and $\log. \frac{1 + h}{1 - h} = \log. \frac{m + 1}{m} = \log. (m + 1)$

$- \log. m = 2 \left(\frac{1}{2m + 1} + \frac{1}{3(2m + 1)^3} + \frac{1}{5(2m + 1)^5} + \text{\&c.}\right)$

Let $m = 1, 2,$ &c. successively, then

$$\log. 2 = 2\left(\frac{1}{3} + \frac{1}{3 \,.\, 3^3} + \frac{1}{5 \,.\, 3^5} + \frac{1}{7 \,.\, 3^7} + \text{\&c.}\right) \quad = .6931472$$

$$\log. 3 = \log. 2 + 2\left(\frac{1}{5} + \frac{1}{3 \,.\, 5^3} + \frac{1}{5 \,.\, 5^5} + \frac{1}{7 \,.\, 5^7} + \text{\&c.}\right) = 1.0986123$$

$$\log. 4 = 2 \log. 2 \quad = 1.3862944$$

(61.) Since if $z = \log._a x$, $\frac{dz}{dx} = \frac{1}{\log._e a \,.\, x}$ (27) $\therefore \log._a x = \frac{1}{\log._e a}$

$\log._e x$, $\log._a$ being *a* log. in a system whose base is a $\therefore$ the log. of any number in that system is found by multiplying the Napierian log. of the number by $\frac{1}{\log._e a} = \mathrm{M}$, which is called the modulus of the system. But a in the common system $= 10$, and $\log._e 10 = 2.3025\ 851$ $\therefore \mathrm{M} = \frac{1}{\log._e a} = .434\ 2944\ 819$. Hence the common log. of any number is found by multiplying its Napierian log. by .434 2944 819.

(62.) Given $f(x)\, f(h) = f(x+h) + f(x-h)$, find the form of $f(x)$.

Let $z = f(x)$, then

$$f(x+h) = z + \frac{dz}{dx}\frac{h}{1} + \frac{d^2z}{dx^2}\frac{h^2}{1 \,.\, 2} + \frac{d^3z}{dx^3}\frac{h^3}{1 \,.\, 2 \,.\, 3} + \frac{d^4z}{dx^4}\frac{h^4}{1 \,.\, 2 \,.\, 3 \,.\, 4} + \text{\&c.}$$

$$\text{And } f(x-h) = z - \frac{dz}{dx}\frac{h}{1} + \frac{d^2z}{dx^2}\frac{h^2}{1 \,.\, 2} - \frac{d^3z}{dx^3}\frac{h^3}{1 \,.\, 2 \,.\, 3} + \frac{d^4z}{dx^4}\frac{h^4}{1 \,.\, 2 \,.\, 3 \,.\, 4} - \text{\&c.}$$

$$\therefore zf(h) = f(x+h) + f(x-h) = 2\left(z + \frac{d^2z}{dx^2}\frac{h^2}{1 \,.\, 2} + \frac{d^4z}{dx^4}\frac{h^4}{1 \,.\, 2 \,.\, 3 \,.\, 4} + \text{\&c.}\right)$$

$$\therefore f(h) = 2\left(1 + \frac{1}{z}\frac{d^2z}{dx^2}\frac{h^2}{1 \,.\, 2} + \frac{1}{z}\frac{d^4z}{dx^4}\frac{h^4}{1 \,.\, 2 \,.\, 3 \,.\, 4} + \text{\&c.}\right)$$

Now since $f(h)$ is independent of x, $\frac{1}{z}\frac{d^2z}{dx^2}$, $\frac{1}{z}\frac{d^4z}{dx^4}$, &c. must be constant. Let $\frac{1}{z}\frac{d^2z}{dx^2} = -a^2 \therefore \frac{d^2z}{dx^2} = -a^2z$, $\frac{d^4z}{dx^4} = -a^2\frac{d^2z}{dx^2} = a^4z$. Hence $f(h) = 2\left(1 - \frac{a^2h^2}{1.2} + \frac{a^4h^4}{1.2.3.4} - \&c.\right) = 2\cos. ah$, and $\therefore f(x) = 2\cos. ax$ and $f(x \pm h) = 2\cos. (ax \pm ah)$. Poisson has founded his proof of the composition of forces on this theorem (*vide Traité de Mechanique, tom.* i. *p.* 151).

(63.) In Taylor's theorem, the increment h may be taken so small that any one term will be greater than the sum of all the terms that succeed it.

Let $z' = z + \frac{dz}{dx}h + \frac{d^2z}{dx^2}\frac{h^2}{1.2} + \frac{d^3z}{dx^3}\frac{h^3}{1.2.3} + \&c.$; then $\frac{z'-z}{h} = \frac{dz}{dx} + \frac{d^2z}{dx^2}\frac{h}{1.2} + \frac{d^3z}{dx^3}\frac{h^2}{1.2.3} + \&c.$

Now when $h = 0$, the right-hand side of this equation becomes $= \frac{dz}{dx}$. It is obvious, therefore, that h may be taken so small that $\frac{dz}{dx} > \frac{d^2z}{dx^2}\frac{h}{1.2} + \frac{d^3z}{dx^3}\frac{h^2}{1.2.3} + \&c.$ Multiplying both sides by h and we have $\frac{dz}{dx}h > \frac{d^2z}{dx^2}\frac{h^2}{1.2} + \frac{d^3z}{dx^3}\frac{h^3}{1.2.3} + \&c.$ In a similar manner it may be proved that any other term is greater than the sum of all the terms that succeed it.

Cases in which Taylor's Theorem fails.

(64.) In general, when $f(x)$ contains a radical, $f(x+h)$ will contain the same radical, because $x+h$ enters wherever x entered. But this will not always be the case when a particular value a is given to x.

Thus if $f(x) = ax^2 + bx^{\frac{3}{2}}$

$$f(x+h) = a(x+h)^2 + b(x+h)^{\frac{3}{2}}$$

But if $f(x) = ax^2 + b(x-a)^{\frac{3}{2}}$, then

$$f(x+h) = a(x+h)^2 + b(x+h-a)^{\frac{3}{2}}$$

Let x become $= a$, then $f(x)_{x=a} = a^3$ and $f(x+h)_{x=a} = a(a+h)^2 + bh^{\frac{3}{2}}$ $\therefore$ in this last case $f(x+h)_{x=a}$ contains a radical which does not enter into $f(x)_{x=a}$.

Again, let $f(x)_{x=\frac{\pi}{2}} = (\tan. x)_{x=\frac{\pi}{2}}$ then $f(x+h)_{x=\frac{\pi}{2}} = \tan.\left(\frac{\pi}{2}+h\right)$

$$= -\tan.\left(\frac{\pi}{2}-h\right) = -\cot. h = -\frac{\cos. h}{\sin. h} = -$$

$$\frac{1 - \frac{h^2}{1 . 2} + \frac{h^4}{1 . 2 . 3 . 4} - \&c.}{h - \frac{h^3}{1 . 2 . 3} + \frac{h^5}{1 . 2 . 3 . 4 . 5} - \&c.} = -h^{-1} + \frac{h}{3} + \frac{h^3}{3^2 . 5} +$$

$\frac{2h^5}{3^2 . 5 . 7} + \frac{h^7}{3^2 . 5^2 . 7} + \&c.$ Therefore when x becomes $= \frac{\pi}{2}$, one of the exponents of h is negative. In this case $f(x)_{x=\frac{\pi}{2}}$ is infinite.

Again, let $z = f(x) = ax^2 + b(x-a)^{\frac{3}{2}}$

$$\frac{dz}{dx} = 2ax + \frac{3}{2}b(x-a)^{\frac{1}{2}}$$

$$\frac{d^2z}{dx^2} = 2\,a + \frac{3\,b}{4\,(x-a)^{\frac{1}{2}}}.$$

$$\therefore \left(\frac{dz}{dx}\right)_{x=a} = 2\,a^2 \;\&\; \left(\frac{d^2z}{dx^2}\right)_{x=a} = \infty.$$

It appears, therefore, that if, in the expansion of $f(x+h)$, when a particular value is given to x, one of the exponents of h be negative, or if h contain a radical which does not exist in the original function, some of the quantities $f(x)$, $\frac{df(x)}{dx}$, $\frac{d^2f(x)}{dx^2}$ &c. are infinite.

To prove that this holds generally, let $z = f(x)$, then $(z')_{x=a} = f(x+h)_{x=a} = A + Bh + Ch^2 + \ldots + Rh^s + Sh^\sigma + \ldots$ Where σ lies between s and $s+1$, we will prove that $\left(\frac{d^{s+1}z}{dx^{s+1}}\right)_{x=a}$ is infinite. It may be proved, as in (59), that if $z' = f(x+h)$, $\frac{dz'}{dx} = \frac{dz'}{dh}$, $\frac{d^2z'}{dx^2} = \frac{d^2z'}{dh^2}$,

$$\frac{d^3z'}{dx^3} = \frac{d^3z'}{dh^3} \;\ldots\ldots\; \frac{d^nz'}{dx^n} = \frac{d^nz'}{dh^n}$$

Therefore $(z')_{x=a} = A + Bh + Ch^2 + \ldots + Rh^s + Sh^\sigma + \ldots$

$$\left(\frac{dz'}{dx}\right)_{x=a} = B + 2\,Ch + \ldots + s\,Rh^{s-1} + \sigma\,Sh^{\sigma-1} + \ldots$$

$$\left(\frac{d^2z'}{dx^2}\right)_{x=a} = 2\,C + \ldots\; s\,\overline{s-1}\,Rh^{s-2} + \sigma\,\overline{\sigma-1}\,.Sh^{\sigma-2} + \&c.$$

Making $h=0$ we have $(z)_{x=a} = A$, $\left(\frac{dz}{dx}\right)_{x=a} = B$, $\left(\frac{d^2z}{dx^2}\right)_{x=a} = 2\,C$, which determines A, B, C, &c.

It appears also that $\left(\frac{d^sz'}{dx^s}\right)_{x=a} = s\,\overline{s-1}\ldots 3\,.\,2\,.\,1\;R + \sigma\,\overline{\sigma-1}\ldots$

$\sigma - s + 1\, Sh^{\sigma - s}$ $\therefore \left(\frac{d^{s+1}z}{dx^{s+1}}\right)_{x=a} = \sigma\,\overline{\sigma - 1}\,\overline{\sigma - 2}\ldots\overline{\sigma - s}\,Sh^{\sigma - s - 1}$.

But $\sigma < s + 1$ $\therefore$ the exponent of h is negative, and $\therefore \left(\frac{d^{s+1}z}{dx^{s+1}}\right)_{x=a} = \infty$ when $h = 0$. It is obvious also that all the differential coefficients which follow $\left(\frac{d^{s+1}z}{dx^{s+1}}\right)_{x=a}$ are infinite when $h = 0$. Taylor's Theorem will therefore enable us to obtain the true expansion of $f(x + h)_{x=a}$ up to the term containing h^s, after which it will fail. The process in such cases must be carried on by the Binomial Theorem or some other algebraical method.

Ex. Let $z = f(x) = 2ax - x^2 + a\sqrt{x^2 - a^2}$

$$\frac{dz}{dx} = 2(a - x) + \frac{ax}{\sqrt{x^2 - a^2}}$$

$$\frac{d^2z}{dx^2} = -2 + \frac{a}{\sqrt{x^2 - a^2}} - \frac{ax^2}{(x^2 - a^2)^{\frac{3}{2}}}$$

&c. = &c.

Making $x = a$ we have $f(a) = a^2$, $\left(\frac{dz}{dx}\right)_{x=a} = + \frac{1}{0} = \infty$, then all the differential coefficients which follow $\left(\frac{dz}{dx}\right)_{x=a}$ are infinite, and the development of $f(x + h)_{x=a}$ contains necessarily one term at least where h has a fractional exponent.

In fact we have, by substituting $(x + h)_{x=a}$ for x in $f(x)$, $f(x + h)_{x=a}$ $= a^2 - h^2 + a\sqrt{h}\sqrt{2a + h} = a^2 - h^2 + 2^{\frac{1}{2}} a^{\frac{3}{2}} h^{\frac{1}{2}} + \frac{a^{\frac{1}{2}} h^{\frac{3}{2}}}{2 \cdot 2^{\frac{1}{2}}} - \frac{h^{\frac{5}{2}}}{16 \cdot 2^{\frac{1}{2}} a^{\frac{1}{2}}} + \frac{h^{\frac{7}{2}}}{64 \cdot 2^{\frac{1}{2}} a^{\frac{3}{2}}} -$ &c., which is the true expansion found by the Binomial Theorem.

For more ample information on the cases in which Taylor's Theorem fails, *vide* Lagrange's Calcul. des Fonctions, p. 69.

Examples on Expansion by Maclaurin's Theorem, Taylor's Theorem, &c.

(1.) Prove that $\left(1-\frac{x}{2}\right)^{-2} = 1 + x + \frac{3}{4}x^2 + \frac{1}{2}x^3 + \frac{5}{16}x^4 + \&c.$

(2.) Prove that $\tan^{-1}\frac{x}{a} = \frac{x}{a} - \frac{x^3}{3a^3} + \frac{x^5}{5a^5} - \frac{x^7}{7a^7} + \&c.$

(3.) Prove that $\cos^5 x = 1 - \frac{5x^2}{1.2} + \frac{65x^4}{1.2.3.4} - \&c.$

(4.) Prove that $e^{\sin^{-1}x} = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{2x^3}{1.2.3} + \frac{5x^4}{1.2.3.4.5} + \&c.$

(5.) Prove that $\sec x = 1 + \frac{x^2}{1.2} + \frac{5x^4}{1.2.3.4} + \&c.$

(6.) Prove that if $y = e^{\frac{ax}{y}}$, then $y = 1 + ax - \frac{a^2x^2}{1.2} + \frac{2^2a^3x^3}{1.2.3} - \&c.$

(7.) Prove that $\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{3^2.5} - \frac{2x^5}{3^3.5.7} - \frac{x^7}{3^3.5^2.7} - \&c.$

(8.) Prove that $\tan(x+h) = \tan x + \sec^2 x . h + 2\sec^2 x \tan x \frac{h^2}{1.2} + 2\sec^2 x(1+3\tan^2 x)\frac{h^3}{1.2.3} + \&c.$

(9.) Prove that $\tan^{-1}(x+h) = \tan^{-1}x + \frac{1}{1+x^2}\frac{h}{1} - \frac{2x}{(1+x^2)^2}\frac{h^2}{1.2} + \frac{2(3x^2-1)}{(1+x^2)^3}\frac{h^3}{1.2.3} \&c.$

(10.) Prove that $e^{a+bx+cx^2+dx^3+\&c.} = e^a\left(1 + bx + \frac{b^2+2c}{1.2}x^2 + \frac{b^3+6bc+6d}{1.2.3}x^3 + \&c.\right)$

CHAPTER V.

VANISHING FRACTIONS.

(65.) If $\frac{F(x)}{f(x)}$ assume the form $\frac{0}{0}$ when a particular value a is given to x, it is evident that each of the terms contains a factor of the form $(x - a)^m$ where m is whole or fractional.

Let $\frac{F(x)}{f(x)} = \frac{P(x-a)^m}{Q(x-a)^n} = \frac{P}{Q}(x-a)^{m-n}$

First, let m and n be integers, then if $m = n$, $\frac{F(x)}{f(x)} = \frac{P}{Q}$ when $x = a$.

If $m > n$, $\frac{F(x)}{f(x)} = 0$ when $x = a$, and if $m < n$, $\frac{F(x)}{f(x)} = \infty$, when $x = a$.

When m and n are integers, the value of $\frac{P_{x=a}}{Q_{x=a}}$ may be found by differentiation.

$F(x) = P(x-a)^m$

$$\frac{d\,F(x)}{dx} = (x-a)^m \frac{dP}{dx} + m(x-a)^{m-1}P$$

$$\frac{d^2\,F(x)}{dx^2} = (x-a)^m \frac{d^2P}{dx^2} + 2m(x-a)^{m-1}\frac{dP}{dx} + m\,\overline{m-1}\,(x-a)^{m-2}P$$

$$\ldots = \ldots\ldots\ldots\ldots$$

$$\frac{d^r\,F(x)}{dx^r} = X(x-a)^m + X'(x-a)^{m-1} + X''(x-a)^{m-2} + \dots$$

$$m\,\overline{m-1}\,\dots\,\overline{m-r+1}\,P\,(x-a)^{m-r}.$$

In a similar manner it appears that

$$\frac{d^r f(x)}{dx^r} = Z(x-a)^n + Z'(x-a)^{n-1} + Z''(x-a)^{n-2} + \dots +$$

$$n\,\overline{n-1}\,\dots\,\overline{n-r+1}\,Q\,(x-a)^{n-r}$$

$$\therefore \frac{\dfrac{d^r\,F(x)}{dx^r}}{\dfrac{d^r f(x)}{dx^r}} = \frac{X(x-a)^m + X'(x-a)^{m-1} + X''(x-a)^{m-2} + \dots}{Z(x-a)^n + Z'(x-a)^{n-1} + Z''(x-a)^{n-2} + \dots}$$

$$\frac{+\,m\,\overline{m-1}\,\dots\,\overline{m-r+1}\,P\,(x-a)^{m-r}}{+\,n\,\overline{n-1}\,\dots\,\overline{n-r+1}\,Q\,(x-a)^{n-r}}$$

Let (1) $m = r$, $m = n$, and $x = a$, then

$$\frac{\dfrac{d^m\,F\,x}{dx^m}}{\dfrac{d^m f(x)}{dx^m}} = \frac{P}{Q} = \frac{F(x)}{f(x)}$$

Let (2) $n = r$, $m > n$, and $x = a$, then

$$\frac{\dfrac{d^n\,F(x)}{dx^n}}{\dfrac{d^n f(x)}{dx^n}} = \frac{0}{n\,\overline{n-1}\,\dots\,\overline{n-n+1}\,Q\,(x-a)^0} = \frac{0}{Q} = 0.$$

Let (3) $m = r$, $m < n$, and $x = a$, then

$$\frac{\dfrac{d^m F(x)}{dx^m}}{\dfrac{d^m f(x)}{dx^m}} = \frac{m\,\overline{m-1}\ldots\overline{m-m+1}\,P}{0} = \infty.$$

(66.) Secondly, when the exponent of the factor $x - a$ in either $F(x)$ or $f(x)$ is fractional, it is obvious that it cannot be reduced to nothing by differentiation, and therefore the above rule is not applicable.

For let $F(x) = P(x-a)^m$, and let m be greater than k and less than $k+1$, then

$$\frac{d^k F(x)}{dx^k} = X(x-a)^m + X'(x-a)^{m-1} + \ldots + m\,\overline{m-1}\ldots$$
$$\overline{m-k+1}\,P(x-a)^{m-k}$$

$$\frac{d^{k+1} F(x)}{dx^{k+1}} = X_1(x-a)^m + X_1'(x-a)^{m-1} + \ldots + m\,\overline{m-1}\ldots$$
$$\overline{m-k}\,P(x-a)^{m-k-1}$$

When $x = a$, $\dfrac{d^k F(x)}{dx^k} = 0$, and $\dfrac{d^{k+1} F(x)}{dx^{k+1}} = \infty$.

Let us substitute $a + h$ for x in the expansions for $F(x)$ and $f(x)$, then

$$\begin{aligned} F(x) &= P h^{\alpha} + Q h^{\beta} + R h^{\gamma} + S h^{\delta} + \&c. \\ f(x) &= P' h^{\alpha'} + Q' h^{\beta'} + R' h^{\gamma'} + S' h^{\delta'} + \&c. \end{aligned} \quad \text{where } \alpha, \beta, \gamma, \&c.$$

as also α', β', γ', &c. are arranged in ascending orders, then

$$\frac{F(x)}{f(x)} = \frac{P h^{\alpha-\alpha'} + Q h^{\beta-\alpha'} + R h^{\gamma-\alpha'} + S h^{\delta-\alpha'} + \&c.}{P' + Q' h^{\beta'-\alpha'} + R' h^{\gamma'-\alpha'} + S' h^{\delta'-\alpha'} + \&c.}$$

First, let $\alpha = \alpha'$ and $h = 0$, then

$$\frac{F(x)}{f(x)} = \frac{P}{P'};$$

Secondly, let $\alpha > \alpha'$ and $h = 0$, then

$$\frac{F(x)}{f(x)} = \frac{0}{P} = 0.$$

Thirdly, let $\alpha < \alpha'$ and $h = 0$, then

$$\frac{F(x)}{f(x)} = \frac{P}{0} = \infty.$$

(67.) A fraction of the form $\frac{F(x)}{f(x)}$, which becomes $= \frac{\infty}{\infty}$, when $x = a$ may be made to assume the form $\frac{0}{0}$.

$$\text{For } \frac{F(x)}{f(x)} = \frac{\frac{1}{f(x)}}{\frac{1}{F(x)}} = \frac{\frac{1}{\infty}}{\frac{1}{\infty}} = \frac{0}{0} \text{ when } x = a.$$

(68.) The expression $F(x) f(x)$, which becomes equal to $0 \times \infty$ when $x = a$, may also be converted into the form $\frac{0}{0}$.

$$\text{For } F(x)\, f(x) = \frac{F(x)}{\frac{1}{f(x)}} = \frac{0}{\frac{1}{\infty}} = \frac{0}{0}, \text{ when } x = a.$$

(69.) A function of the form $F(x) - f(x)$, which becomes $= \infty - \infty$ when $x = a$, may be reduced to the form $\frac{0}{0}$.

$$\text{For let } F(x) - f(x) = u - v = \frac{1}{u'} - \frac{1}{v'} = \frac{v' - u'}{u'v'} = \frac{0}{0}, \text{ when } x = a.$$

Ex. (1.) Find the value of $\frac{F(x)}{f(x)} = \frac{a^n - x^n}{a - x} = \frac{0}{0}$, when $x = a$

$$\frac{d\,F(x)}{dx} = -\,n\,x^{n-1} \text{ and } \frac{d\,f(x)}{dx} = -1 \;\therefore\; \frac{F(x)}{f(x)} = \frac{\dfrac{d\,F(x)}{dx}}{\dfrac{d\,f(x)}{dx}} = n\,a^{n-1}$$

when $x = a$.

Ex. 2. Find the value $\dfrac{F(x)}{f(x)} = \dfrac{a^x - b^x}{x} = \dfrac{0}{0}$ when $x = 0$

$$\frac{d\,F(x)}{dx} = a^x \log.\, a - b^x \log.\, b \text{ and } \frac{d\,f(x)}{dx} = 1 \;\therefore\; \frac{F(x)}{f(x)} = \frac{\dfrac{d\,F(x)}{dx}}{\dfrac{d\,f(x)}{dx}} =$$

$\log.\, a - \log.\, b = \log.\, \dfrac{a}{b}$ when $x = 0$.

The same result may be obtained without differentiation, as follows. Thus—

$$a^x = 1 + Ax + \frac{A^2 x^2}{1\,.\,2} + \&c. \quad (25)$$

$$b^x = 1 + A'x + \frac{A'^2 x^2}{1\,.\,2} + \&c.$$

$$\therefore\; a^x - b^x = (A - A')\,x + (A^2 - A'^2)\frac{x^2}{1\,.\,2} + \&c.$$

$$\therefore\; \frac{F(x)}{f(x)} = \frac{a^x - b^x}{x} = (A - A') \text{ when } x = 0 = \log.\, a - \log.\, b \; (26)$$

$= \log.\, \dfrac{a}{b}$ as before.

Ex. (3.) Find the value of $\dfrac{F(x)}{f(x)} = \dfrac{a^{\log.\, x} - x}{\log.\, x} = \dfrac{0}{0}$ when $x = 1$.

$$\frac{d\,F(x)}{dx} = \frac{\log.\, a\,.\, a^{\log.\, x}}{x} - 1 \text{ and } \frac{d\,f(x)}{dx} = \frac{1}{x} \;\therefore\; \frac{F(x)}{f(x)} = \frac{\dfrac{d\,Fx}{dx}}{\dfrac{d\,f(x)}{dx}} =$$

$$\frac{\dfrac{\log. a \,.\, a^{\log. x}}{x} - 1}{\dfrac{1}{x}} = \log. a \,.\, a^{\log. x} - x = \log. a - 1 \text{ when } x = 1.$$

Ex. (4.) Find the value of $\dfrac{F(x)}{f(x)} = \dfrac{\cos. x - \cos. 2x}{\cos. x - \cos. 3x} = \dfrac{0}{0}$ when $x = 0$.

$$\frac{d\,F(x)}{dx} = -\sin. x + 2\sin. 2x, \quad \frac{d\,f(x)}{dx} = -\sin. x + 3\sin. 3x$$

$$\frac{d^2 F(x)}{dx^2} = -\cos. x + 4\cos. 2x, \quad \frac{d^2 f(x)}{dx^2} = -\cos. x + 9\cos. 3x$$

$$\therefore \frac{F(x)}{f(x)} = \frac{\dfrac{d^2 F(x)}{dx^2}}{\dfrac{d^2 f(x)}{dx^2}} = \frac{-\cos. x + 4\cos. 2x}{-\cos. x + 9\cos. 3x} = \frac{3}{8} \text{ when } x = 0.$$

Ex. (5.) Find the value of $\dfrac{F(\theta)}{f(\theta)} = \dfrac{\sin.^2 m\theta}{\sin.^2 \theta} = \dfrac{0}{0}$ when $\theta = 0$.

$$\frac{d\,F(\theta)}{d\theta} = 2m\sin. m\theta \cos. m\theta \qquad \frac{d\,f(\theta)}{d\theta} = 2\sin. \theta \cos. \theta$$

$$\frac{d^2 F(\theta)}{d\theta^2} = 2m^2\cos.^2 m\theta - 2m^2\sin.^2 m\theta, \quad \frac{d^2 f\theta}{d\theta^2} = 2\cos.^2\theta - 2\sin.^2\theta$$

$$\therefore \frac{F(\theta)}{f(\theta)} = \frac{\dfrac{d^2 F(\theta)}{d\theta^2}}{\dfrac{d^2 f(\theta)}{d\theta^2}} = \frac{2m^2\cos.^2 m\theta - 2m^2\sin.^2\theta}{2\cos.^2\theta - 2\sin.^2\theta} = m^2 \text{ when } \theta = 0.$$

(Vide *Airy's Undulatory Theory of Optics, Article* 84.)

Ex. (6.) Find the value of $\dfrac{F(x)}{f(x)} = \dfrac{(x^2 - a^2)^{\frac{3}{2}}}{(x^3 - a^3)^{\frac{3}{2}}} = \dfrac{0}{0}$ when $x = a$.

Let $x = a + h$, then

$$(x^2 - a^2)^{\frac{1}{2}} = (2\,ah + h^2)^{\frac{1}{2}} = (2\,ah)^{\frac{1}{2}}\left(1 + \tfrac{1}{4}\,\frac{h}{a} + \&c.\right)$$

and $(x^3 - a^3)^{\frac{1}{2}} = (3\,a^2h + 3\,ah^2 + h^3)^{\frac{1}{2}} = (3\,a^2h)^{\frac{1}{2}}\left(1 + \tfrac{1}{2}\,\frac{h}{a} + \&c.\right)$

$$\therefore \frac{\mathrm{F}(x)}{f(x)} = \frac{(2\,ah)^{\frac{1}{2}}\left(1 + \frac{1}{4}\,\frac{h}{a} + \&c.\right)}{(3\,a^2h)^{\frac{1}{2}}\left(1 + \frac{1}{2}\,\frac{h}{a} + \&c.\right)} = \left(\frac{2}{3a}\right)^{\frac{1}{2}} \frac{1 + \frac{1}{4}\,\frac{h}{a} + \&c.}{1 + \frac{1}{2}\,\frac{h}{a} + \&c.}$$

when $x = a + h = \left(\dfrac{2}{3\,a}\right)^{\frac{1}{2}}$ when $h = 0$.

Examples for Practice.

(1.) Find the value of $\dfrac{1 - 4x^2 + 3x^3}{1 - 6x^2 + 5x^3}$ when $x = 1$. Ans. $\dfrac{1}{3}$.

(2.) Find the value of $\dfrac{x^3 - 1}{x^3 + 2x^2 - x - 2}$ when $x = 1$. Ans. $\dfrac{1}{2}$.

(3.) Find the value of $\dfrac{x^3 - ax^2 + a^2x - a^3}{x^2 - 2ax + a^2}$ when $x = a$. Ans. $2a$

(4.) Find the value of $\dfrac{a(x^2 + c^2) - 2acx}{b(x^2 + c^2) - 2bcx}$ when $x = c$. Ans. $\dfrac{a}{b}$.

(5.) Find the value of $\dfrac{\tan. x - \sin. x}{\sin.^3 x}$ when $x = 0$. Ans. $\dfrac{1}{2}$.

(6.) Find the value of $\dfrac{x}{x-1} - \dfrac{1}{\log. x}$ when $x = 1$. Ans. $\dfrac{1}{2}$.

(7.) Find the value of $\dfrac{e^x - 1 - \log.(1 + x)}{x^2}$ when $x = 0$. Ans. 1.

(8.) Find the value of $\dfrac{a^{\log x} - x}{\log. x}$ when $x = 1$. Ans. $\log.\left(\dfrac{a}{e}\right)$.

(9.) Find the value of $\frac{1}{\log. (1 + x)} - \frac{1}{x}$ when $x = 0$. Ans. $\frac{1}{2}$.

(10.) Find the value of $\frac{\log. \tan. x}{\log. \tan. 2x}$ when $x = 0$. Ans. 1.

(11.) Find the value of $\frac{1 - x + \log. x}{1 - \sqrt{2x - x^2}}$ when $x = 1$. Ans. ± 1.

(12.) Find the value of $2^x \tan. \frac{a}{2^x}$ when $x = \infty$. Ans. a.

(13.) Find the value of $(\sin. x)^{\sin x}$ when $x = 0$. Ans. 1.

(14.) Find the value of $(\cot. x)^{\sin x}$ when $x = 0$. Ans. 1.

(15.) Find the value of $\frac{\pi}{4x} \tan. \frac{\pi x}{2}$ when $x = 0$. Ans. $\frac{\pi^2}{8}$.

(16.) Find the value of $\left(\sec \frac{\pi x}{2}\right)^2$ vers. $2\pi x$ when $x = 1$. Ans. 8

(17.) Find the value of $\frac{a(1 - x)}{\cot \frac{\pi x}{2}}$ when $x = 1$. Ans. $\frac{2a}{\pi}$.

(18.) Find the value of $\frac{e^x - e^{\sin x}}{x - \sin. x}$ when $x = 0$. Ans. 1.

(19.) Find the value of $\frac{x^2 - a^2}{x^2}$ tan. $\frac{\pi x}{2a}$ when $x = a$. Ans. $-\frac{4}{\pi}$.

(20.) Find the value of $(\cos. \alpha x)^{(\operatorname{cosec} \beta x)^2}$ when $x = 0$. Ans. $e^{-\frac{a^2}{2\beta^2}}$.

CHAPTER VI.

MAXIMA AND MINIMA OF FUNCTIONS OF ONE VARIABLE.

(70.) If a quantity first increase and then decrease, its greatest value is called a maximum; and if it first decrease and then increase, its least value is called a minimum.

Thus, in a circle, if an arc increase from 0 to 90°, its sine will increase from 0 to radius, and if the arc increase from 90 to 180°, the sine will diminish from radius to 0. The sine of 90° is therefore a maximum.

Again, the line drawn from the focus of a parabola to the vertex is less than any other line drawn from the same point to the curve, it is therefore a minimum.

(71.) Let $z = f(x)$, and let $f(a)$ be greater than either $f(a + h)$ or $f(a - h)$, then $f(a)$ is a maximum; but if $f(a)$ be less than either $f(a + h)$ or $f(a - h)$, $f(a)$ is a minimum.

(72.) When $z = f(x)$ is a maximum or minimum, $\frac{dz}{dx} = 0$.

$$\text{For } f(x + h) = z + \frac{dz}{dx} h + \frac{d^2z}{dx^2} \frac{h^2}{1 \cdot 2} + \frac{d^3z}{dx^3} \frac{h^3}{1 \cdot 2 \cdot 3} + \&c.$$

$$\text{and } f(x - h) = z - \frac{dz}{dx} h + \frac{d^2z}{dx^2} \frac{h^2}{1 \cdot 2} - \frac{d^3z}{dx^3} \frac{h^3}{1 \cdot 2 \cdot 3} + \&c.$$

But (63.) h may be taken so small that $\frac{dz}{dx} h$ will be greater than the sum of all the terms that follow it. Consequently $f(x + h)$ is greater than $f(x)$, and $f(x - h)$ is less than it. $\therefore f(x)$ is neither a maximum nor minimum, unless $\frac{dz}{dx} h = 0$, that is $\frac{dz}{dx} = 0$. $\therefore$ when $f(x)$

is a maximum or minimum $f(x+h) = z + \frac{d^2z}{dx^2}\frac{h^2}{1.2} + \frac{d^3z}{dx^3}\frac{h^3}{1.2.3}$ + &c.

and $f(x-h) = z + \frac{d^2z}{dx^2}\frac{h^2}{1.2} - \frac{d^3z}{dx^3}\frac{h^3}{1.2.3}$ + &c.

In this case also h may be taken so small that $\frac{d^2z}{dx^2}\frac{h^2}{1.2}$ will be greater than the sum of all the terms that follow it. But as $\frac{d^2z}{dx^2}\frac{h^2}{1.2}$ has the same sign in both developments, it follows that if $\frac{d^2z}{dx^2}$ be positive, $f(x+h)$ and $f(x-h)$ are both greater than $f(x)$. $\therefore f(x)$ is a minimum; but if $\frac{d^2z}{dx^2}$ be negative, $f(x)$ is a maximum.

(73.) If both $\frac{dz}{dx}$ and $\frac{d^2z}{dx^2}$ vanish in the developments of $f(x+h)$ and $f(x-h)$, $\frac{d^3z}{dx^3}$ must also vanish, in order that $f(x)$ may be a maximum or minimum Then taking h very small, $\frac{d^4z}{dx^4}\frac{h^4}{1.2.3.4}$ will be greater than the sum of all the terms that follow it It appears, therefore, that in this case $f(x)$ will be a minimum when $\frac{d^4z}{dx^4}$ is positive, and a maximum when it is negative. In general, if the first co-efficient which does not vanish be even, the function will be a minimum when its sign is positive, and a maximum when it is negative.

(74.) If z be a maximum or minimum, mz is also a maximum or minimum, m being any positive number.

For since z is a maximum or minimum, $\frac{dz}{dx} = 0 \quad \therefore m\frac{dz}{dx} = 0$, and mz is a maximum or minimum

(75.) If z be a maximum or minimum, z^n is also a maximum or minimum, n being any positive integer.

For since z is a maximum or minimum, $\frac{dz}{dx} = 0 \therefore nz^{n-1}\frac{dz}{dx} = 0$, and $\therefore z^n$ is a maximum or minimum.

(76.) If z be a maximum, $\frac{1}{z}$ is a minimum, and conversely.

For let $u = \frac{1}{z}$, then $\frac{du}{dx} = -\frac{1}{z^2}\frac{dz}{dx}$, $\frac{d^2u}{dx^2} = \frac{2}{z^3}\frac{dz^2}{dx^2} - \frac{1}{z^2}\frac{d^2z}{dx^2}$

$= -\frac{1}{z^2}\frac{d^2z}{dx^2}$ when z is a maximum. $\therefore$ if $\frac{d^2z}{dx^2}$ be negative, $\frac{d^2u}{dx^2}$ is positive. $\therefore$ when z is a maximum, $\frac{1}{z}$ is a minimum.

(77.) If z be a maximum or minimum, log. z will generally be a maximum or minimum.

For since z is a maximum or minimum, $\frac{dz}{dx} = 0$, $\therefore \frac{1}{z}\frac{dz}{dx} = 0$, and

$\therefore$ log. z is a maximum or minimum, unless when $z = 0$, and $x = a$, in which case $\frac{1}{z}\frac{dz}{dx}$ becomes of the form $\infty . 0$ or $\frac{0}{0}$, which is indeterminate.

Ex. 1. Let $z = a - (b - x)^2$

$$\frac{dz}{dx} = 2(b - x) = 0 \therefore x = b$$

$$\frac{d^2z}{dx^2} = -2 \therefore \text{ when } x = b, z \text{ is a maximum.}$$

Ex. 2. To divide a into two such parts that the mth power of the one, multiplied by the nth power of the other, may be a maximum.

Let x = the one part, then $a - x$ = the other $\therefore z = x^m (a - x)^n$

$\frac{dz}{dx} = m x^{m-1}(a - x)^n - n x^m (a - x)^{n-1} = 0 \therefore m(a - x) - n x = 0$

$\therefore x = \frac{m\,a}{m+n}$, $\frac{d^2z}{dx^2} = -\frac{m^{m-1}\,n^{n-1}\,a^{m+n-2}}{(m+n)^{m+n-2}} \therefore \frac{m\,a}{m+n}$ when substituted for x renders the function a maximum.

Ex. 3. Let $z = \frac{ax}{a^2+x^2}$

$$\frac{dz}{dx} = \frac{a\,(a^2 - x^2)}{(a^2+x^2)^2} = 0 \therefore x = \pm a$$

$\frac{d^2z}{dx^2} = -\frac{1}{2\,a^2}$ when $x = +a$, and $\frac{d^2z}{dx^2} = \frac{1}{2\,a^2}$ when $x = -a \therefore$ $x = +a$ renders the function a maximum, and $x = -a$ renders it a minimum.

Ex. 4. Let $z = \sin. x + \cos. x$

$$\therefore \frac{dz}{dx} = \cos. x - \sin. x = 0 \therefore \sin. x = \cos. x, \text{ and } \therefore x = 45^\circ.$$

Again, $\frac{d^2z}{dx^2} = -\sin. x - \cos. x = -\sin. 45^\circ - \cos. 45^\circ = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2} \therefore x = 45^\circ$ renders the function a maximum.

Ex. 5. Let $z = m \sin. (x - a) \cos. x$,

$$\frac{dz}{dx} = m \cos. (x - a) \cos. x - m \sin. (x - a) \sin. x = 0$$

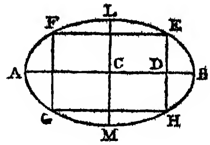
$$\cos. (2\,x - a) = 0 \therefore 2\,x - a = \pm \frac{\pi}{2} \therefore x = \frac{a}{2} \pm \frac{\pi}{4}$$

$$\therefore z = m \sin. \left(\frac{\pi}{4} - \frac{a}{2}\right) \cos. \left(\frac{a}{2} + \frac{\pi}{4}\right) = m \left(\sin. \frac{\pi}{4} \cos. \frac{a}{2} - \cos. \frac{\pi}{4} \sin. \frac{a}{2}\right)$$

$$\times \left(\cos.\frac{\pi}{4}\cos.\frac{a}{2} - \sin.\frac{\pi}{4}\sin.\frac{a}{2}\right) = \frac{m}{2}\left(\cos.^2\frac{a}{2} + \sin.^2\frac{a}{2} - \sin.\,a\right)$$

$$= \frac{m}{2}\,(1 - \sin.\,a) \text{ a maximum, or } z = \frac{m}{2}\,(1 + \sin.\,a) \text{ a minimum.}$$

Ex. 6. To inscribe the greatest rectangle in a given ellipse—



Let A E B H be an ellipse, C its centre, A B the transverse, and L M the conjugate axis.

Let A C $= a$, C L $= b$, C D $= x$, and E D $= y$, then $y = \frac{b}{a}\sqrt{a^2 - x^2}$.

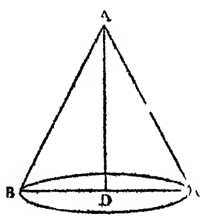
But the area of the rectangle $= 4$ C D . E D $= \frac{4\,b\,x}{a}\sqrt{a^2 - x^2}$

which will be a maximum when $x\sqrt{a^2 - x^2}$ is a maximum (74)

$$\therefore \frac{dz}{dx} = (a^2 - x^2)^{\frac{1}{2}} - \frac{x^2}{(a^2 - x^2)^{\frac{1}{2}}} = 0 \therefore x = \frac{a}{2}\sqrt{2}, \text{ when the area}$$

of the rectangle is a maximum.

Ex. 7. To determine an upright cone that has the greatest solidity with a given surface—



Let B D $= x$ and A B $= y$.

then $\pi x^2 + \pi x y = a \therefore y = \frac{a - \pi x^2}{\pi x}$.

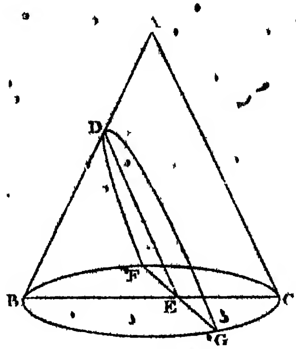
But A D^2 = A B^2 − B D^2 $= y^2 - x^2$

$$= \frac{a^2 - 2\,a\,\pi x^2}{\pi^2 x^2} \therefore S = \frac{\pi x^2}{3}\left(\frac{a^2 - 2\,a\,\pi x^2}{\pi^2 x^2}\right)^{\frac{1}{2}} = \frac{1}{3}\,(a^2 x^2 - 2\,a\,\pi x^4)^{\frac{1}{2}},$$

which will be a maximum when $a x^2 - 2\,\pi x^4$ is a maximum (74) and

$$(75) \therefore \frac{dz}{dx} = 2\,a\,x - 8\,\pi x^3 = 0, \therefore x = \sqrt{\frac{a}{4\,\pi}}$$

Ex. 8. To determine the greatest parabola that can be formed by cutting a given upright cone—



Let $B E = x$, $B C = 2 r$, $A C = b$, and $F E = y$,

then since $B E : E D :: B C : A C$

$$x : E D :: 2 r : b$$

$$\therefore E D = \frac{b x}{2 r}.$$

But $y = \sqrt{2 r x - x^2}$ $\therefore$ the area

$= \frac{4 F E . D E}{3} = \frac{2 b x}{3 r}\sqrt{2 r x - x^2}$ which will be a maximum when

$2 r x^3 - x^4$ is a maximum (74) and (75) $\therefore \frac{dz}{dx} = 6 r x^2 - 4 x^3 = 0$

$\therefore x = \frac{3 r}{2}$.

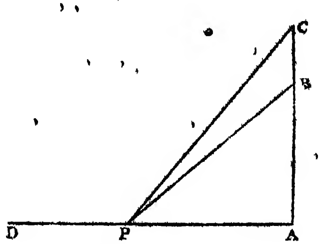
Ex. 9. Let $z = x^{\frac{1}{x}}$, find z when it is a maximum or minimum;

$\frac{dz}{dx} = x^{\frac{1}{x}}\left(\frac{1 - \log. x}{x^2}\right) = 0 \therefore 1 - \log. x = 0$, or $\log. x = 1 \therefore x = e$, and

$z = e^{\frac{1}{e}}$ a maximum.

Ex. 10. To find a point in the straight line A D, at which A B subtends the greatest angle; A B C being perpendicular to A D.

When the angle is a maximum its tangent is a maximum.



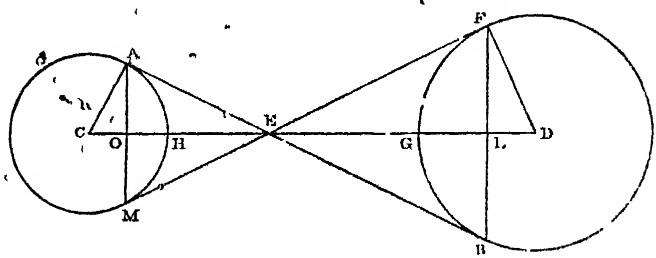
Let P be the point $A P = x$, $A B = b$, $A C = a$; tan. $B P C =$ tan. $(A P C - A P B) =$
$\frac{\text{tan. } A P C - \text{tan. } A P B}{1 + \text{tan. } A P C \text{ tan. } A P B} =$

$$\frac{\frac{a}{x} - \frac{b}{x}}{1 + \frac{ab}{x^2}} = \frac{(a-b)x}{ab + x^2} \therefore \frac{dz}{dx} = ab + x^2 - 2x^2 = 0 \therefore x = \sqrt{ab}$$

$\therefore$ the circle described about B P C will touch A D in P.

Ex. 11. To find a point in the line joining the centres of two spheres, from which the greatest portion of spherical surface is visible.

Let C D $= a$, D F $= r$, E D $= x$ $\therefore$ C E $= a - x$, A C $= r'$.



E D : D F : : D F : D L

$$x : r \quad :: \quad r : \text{D L} \quad \therefore \text{D L} = \frac{r^2}{x}$$

Hence G L $= r - \frac{r^2}{x} = \frac{rx - r^2}{x}$ $\therefore$ the visible area of F G B $=$

$2\pi r \frac{rx - r^2}{x}$, and of A H M $= 2\pi r' \frac{r'(a-x) - r'^2}{a-x}$ $\therefore$ the whole

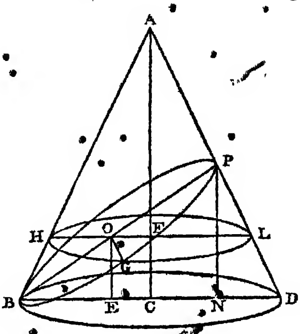
visible area $= 2\pi r \frac{rx - r^2}{x} + 2\pi r' \frac{r'(a-x) - r'^2}{a-x}$, which will be

a maximum when $r^2 - \frac{r^3}{x} + r'^2 - \frac{r'^3}{a-x}$ is a maximum $\therefore \frac{dz}{dx} =$

$$\frac{r^3}{x^2} - \frac{r'^3}{(a-x)^2} = 0 \quad \therefore \quad r^3 (a-x)^2 = r'^3 x^2 \quad \therefore \quad r^{\frac{3}{2}} (a-x) = r'^{\frac{3}{2}} x$$

$$\therefore x = \frac{a r^{\frac{3}{2}}}{r^{\frac{3}{2}} + r'^{\frac{3}{2}}}$$

Ex. 12. To determine the greatest ellipse that can be cut from a given cone.



Let A B D be the cone, and B G P the ellipse, A C $= a$, D C $= b$, C N $= x$, and N P $= y$, then $a : b :: y : b - x \therefore y = \frac{a(b-x)}{b} \therefore \frac{\text{B P}}{2} = \frac{1}{2}\sqrt{\text{B N}^2 + \text{N P}^2} = \frac{1}{2}\sqrt{(b+x)^2 + \frac{a^2}{b^2}(b-x)^2}$, B O $= \frac{\text{B P}}{2}$, E O $= \frac{a(b-x)}{2b}$, A F $= \frac{a(b+x)}{2b}$,

B E $= \frac{1}{2}$ B N $= \frac{1}{2}(b+x) \therefore$ E C $= \frac{1}{2}(b-x) =$ O F,

$\frac{a(b+x)}{2b}$: F L :: $a : b \therefore$ F L $= \frac{b+x}{2}$, and C L $= b$, and H O $= x$,

O G $= \sqrt{bx} \therefore$ the area of the ellipse $= \frac{\pi\sqrt{bx}}{2}\sqrt{(b+x)^2 + \frac{a^2}{b^2}(b-x)^2}$, which will be a maximum when $x^{\frac{1}{2}}(b^2(b+x)^2 + a^2(b-x)^2)^{\frac{1}{2}}$ is a maximum $\therefore \frac{dz}{dx} = \frac{1}{2}x^{-\frac{1}{2}}(b^2(b+x)^2 + a^2(b-x)^2)^{\frac{1}{2}} + \frac{1}{2}x^{\frac{1}{2}}$

$(b^2(b+x)^2 + a^2(b-x))^{-\frac{1}{2}}(2b^2(b+x) - 2a^2(b-x)) = 0.$

$$\therefore b^2(b+x)^2 + a^2(b-x)^2 + x(2b^2(b+x) - 2a^2(b-x)) = 0$$

$$3(a^2+b^2)x^2 - 4(a^2-b^2)bx = -(a^2+b^2)b^2$$

$$x^2 - \frac{4(a^2-b^2)}{3(a^2+b^2)}bx = -\frac{b^2}{3}$$

$$= \frac{2\, b\,(a^2 - b^2) \pm b(a^4 - 14\, a^2 b^2 + b^4)^{\frac{1}{2}}}{3\,(a^2 + b^2)}$$

which is possible when $a^4 + b^4 > 14\ a^2 b^2$, or $a^2 + b^2 > 4\ ab$, or

$a > b\,(2 + \sqrt{3})$, or $\frac{b}{a} = \tan.\ \frac{1}{2}\ \text{A} < \frac{1}{2 + \sqrt{3}} < .2679$ $\therefore$ A must be

less than 31° .. 5′.

EXAMPLES FOR PRACTICE.

(1.) Let $z = \frac{\log.\ x}{x^n}$, find the value of z when it is a maximum.—

Ans. $x = e^{\frac{1}{n}}$, and $z = \frac{1}{ne}$.

(2.) Let $z = (a^2 + c^2 - 2\,cx)^{\frac{1}{2}} + x$, find the value of z when it is

a maximum or minimum.—Ans. $x = \frac{a^2}{2\,c}$, and $z = \frac{a^2 + 2\,c^2}{2\,c}$.

(3.) Let $z = x^2\,(a - x)^3$, find the value of z when it is a maximum.

—Ans. $x = \frac{2\,a}{5}$, and $z = \frac{108\,a^5}{3125}$.

(4.) Let $z = \left(\frac{a}{x}\right)^x$, find z when it is a maximum.—Ans. $x = \frac{a}{e}$,

and $z = e^{\frac{a}{e}}$.

(5.) Let $z = \frac{x}{\log.\ x}$, find z when it is a maximum or minimum.—

Ans. $x = e$, and $z = e$ a minimum.

(6.) Let $z = \dfrac{(a^2 x - x^3)^{\frac{1}{2}}}{2^{\frac{1}{2}} a^{\frac{3}{2}} + x^{\frac{3}{2}}}$, find z when it is a maximum or minimum.—Ans. $x = \dfrac{a}{2}$, and $z = \dfrac{1}{3^{\frac{3}{4}}}$ a maximum.

(7.) Let $z = \sin. x \cos.^2 x$, find z when it is a maximum or minimum. —Ans. when $x = 90^\circ$, $z = 0$, a minimum, and when $x = \cos.^{-1} \sqrt{\dfrac{2}{3}}$, $z = \dfrac{2}{9}\sqrt{3}$ a maximum.

(8.) To inscribe the greatest rectangle in a given parabola.

(9.) To determine the dimensions of the least isosceles triangle that can be described about a given circle.—Ans. The perpendicular altitude $= 3\,r$.

(10.) Through a given point within a given angle, to draw a straight line, so that the sum of the segments intercepted from the vertex of the angle shall be a minimum.

(11.) Draw a tangent to an ellipse, so that the part of it intercepted between the axes produced, shall be a minimum. Let a and b be the semi-axes, x the absciss of the point of contact, the centre being the origin; then $x = \sqrt{\dfrac{a^3}{a+b}}$, $y = \sqrt{\dfrac{b^3}{a+b}}$, and $u = a + b =$ tangent.

(12.) Let $z = \sin.^m x \sin.^n (a - x)$, then $x = \dfrac{1}{2}\left(a - \sin.^{-1}\left(\dfrac{n-m}{n+m} \sin. a\right)\right)$ when z is a maximum.

(13.) A circle and an ellipse have the same major axis, it is required to compare the areas of the greatest rectangles that can be inscribed in them. Let a and b be the major and minor semi-axes of the ellipse, then the rectangles are to one another as $a : b$.

(14.) If the semi-axes of an ellipse be $2\,x'$ and x', it is required to

find the value of x when the area of the ellipse is a minimum. Let e be the base of the Napierian system of log., then $x = \frac{1}{e}$.

(15.) Of all right cones having a given volume, determine that whose surface is a maximum. Let x = the height, y the radius of the base, and $\frac{\pi a^3}{3}$ the volume, then $x = 2a$, $y = \frac{a}{\sqrt{2}}$, and the area $= 2 \pi a^2$.

CHAPTER VII.

APPLICATION OF TAYLOR'S THEOREM TO THE DEVELOPMENT OF FUNCTIONS OF TWO OR MORE VARIABLES.

(78.) Given $z = f(x, y)$ to find $z' = f(x + h, y + k)$, where h and k are of any magnitudes.

First, let x become $= x + h$, and y remain constant, then $z =$

$$f(x+h, y) = z + \frac{dz}{dx} h + \frac{d^2z}{dx^2} \frac{h^2}{1.2} + \frac{d^3z}{dx^3} \frac{h^3}{1.2.3} + \&c. \quad (1.)$$

But $z = f(x, y)$ $\therefore \frac{dz}{dx}, \frac{d^2z}{dx^2}, \frac{d^3z}{dx^3}$, &c. are also functions of x and y.

Let y now become equal to $y + k$, then z must be replaced in (1) by

$$z + \frac{dz}{dy} k + \frac{d^2z}{dy^2} \frac{k^2}{1.2} + \frac{d^3z}{dy^3} \frac{k^3}{1.2.3} + \&c.$$

$\frac{dz}{dx}$ by $\frac{dz}{dx} + \frac{d.\frac{dz}{dx}}{dy} k + \frac{d^2.\frac{dz}{dx}}{dy^2} \frac{k^2}{1.2} + \frac{d^3.\frac{dz}{dx}}{dy^3} \frac{k^3}{1.2.3} + \&c.$

$$= \frac{dz}{dx} + \frac{d^2z}{dx\,dy} k + \frac{d^3z}{dx\,dy^2} \frac{k^2}{1.2} + \frac{d^4z}{dx\,dy^3} \frac{k^3}{1.2.3} + \&c.$$

$\frac{d^2z}{dx^2}$ by $\frac{d^2z}{dx^2} + \frac{d.\frac{d^2z}{dx^2}}{dy} k + \frac{d^2.\frac{d^2z}{dx^2}}{dy^2} \frac{k^2}{1.2} + \frac{d^3.\frac{d^2z}{dx^2}}{dy^3} \frac{k^3}{1.2.3} + \&c.$

$$= \frac{d^2z}{dx^2} + \frac{d^3z}{dx^2\,dy} k + \frac{d^4z}{dx^2\,dy^2} \frac{k^2}{1.2} + \frac{d^5z}{dx^2\,dy^3} \frac{k^3}{1.2.3} + \&c.$$

&c. &c. &c. &c.

$$\therefore z' = f(x + h, y + k) = z + \frac{dz}{dy} k + \frac{d^2z}{dy^2} \frac{k^2}{1 . 2} + \frac{d^3z}{dy^3} \frac{k^3}{1 . 2 . 3} + \&c.$$

$$+ \frac{dz}{dx} h + \frac{d^2z}{dx\,dy} kh + \frac{d^3z}{dx\,dy^2} \frac{k^2 h}{1 . 2} + \&c.$$

$$+ \frac{d^2z}{dx^2} \frac{h^2}{1 . 2} + \frac{d^3z}{dx^2\,dy} \frac{kh^2}{1 . 2} + \&c.$$

$$+ \frac{d^3z}{dx^3} \frac{h^3}{1 . 2 . 3} + \&c.$$

$$+ \&c.$$

which is Taylor's Theorem when applied to the development of functions of two independent variables, x and y.

(79.) If we had supposed y to become $y + k$, while x remained constant, then

$$z = f(x, y + k) = z + \frac{dz}{dy} k + \frac{d^2z}{dy^2} \frac{k^2}{1 . 2} + \frac{d^3z}{dy^3} \frac{k^3}{1 . 2 . 3} + \&c.$$

Substituting $x + h$ for x in z, $\frac{dz}{dy}$, $\frac{d^2z}{dy^2}$, $\frac{d^3z}{dy^3}$, &c. we have

$$z' = f(x + h, y + k) = z + \frac{dz}{dx} h + \frac{d^2z}{dx^2} \frac{h^2}{1 . 2} + \frac{d^3z}{dx^3} \frac{h^3}{1 . 2 . 3} + \&c.$$

$$+ \frac{dz}{dy} k + \frac{d^2z}{dy\,dx} hk + \frac{d^3z}{dy\,dx^2} \frac{h^2 k}{1 . 2} + \&c.$$

$$+ \frac{d^2z}{dy^2} \frac{k^2}{1 . 2} + \frac{d^3z}{dy^2\,dx} \frac{k^2 h}{1 . 2} + \&c.$$

$$+ \frac{d^3z}{dy^3} \frac{k^3}{1 . 2 . 3} + \&c.$$

$$+ \&c.$$

COR. Since the series must be equal, the coefficients of the same powers and combinations of h and k are equal; hence

$$\frac{d^2z}{dy\,dx} = \frac{d^2z}{dx\,dy}$$

$$\frac{d^3z}{dy\,dx^2} = \frac{d^3z}{dx^2dy}$$

$$\&c. = \&c.$$

$$\frac{d^{m+n}z}{dy^m dx^n} = \frac{d^{m+n}z}{dx^n dy^m}$$

It appears therefore that the order of differentiation is indifferent; or that the differential coefficient of z differentiated m times with respect to y, and then n times with respect to x, is equal to the differential coefficient of z differentiated n times with respect to x and then m times with respect to y.

COR. 2. $\frac{d^3z}{dy\,dx^2} = \frac{d^3z}{dy\,dx\,dx} = d\,.\,\frac{\frac{d^2z}{dy\,dx}}{dx} = d\,.\,\frac{\frac{d^2z}{dx\,dy}}{dx} = \frac{d^3z}{dx\,dy\,dx}$

(80.) $\frac{dz}{dx}$, $\frac{d^2z}{dx^2}$, $\frac{d^3z}{dx^3}$, &c. are the differential coefficients upon the hypothesis that x is the only variable, and $\frac{dz}{dy}$, $\frac{d^2z}{dy^2}$, $\frac{d^3z}{dy^3}$, &c. are those upon the hypothesis that y is the only variable. They are therefore called *partial differential coefficients*, and are usually included within brackets, thus, $\left(\frac{dz}{dx}\right)$ and $\left(\frac{dz}{dy}\right)$ are partial coefficients with respect to x and y respectively. $\left(\frac{dz}{dx}\right)dx$ and $\left(\frac{dz}{dy}\right)dy$ are the *partial differentials* of z with respect to x and y, therefore $dz = \left(\frac{dz}{dx}\right)dx + \left(\frac{dz}{dy}\right)dy$ will be the *total differential* of z.

(81.) If we represent the indeterminate magnitudes h and k by dx and dy, we will have

$$(1)\ dz = \left(\frac{dz}{dx}\right) dx + \left(\frac{dz}{dy}\right) dy.$$

$$(2)\ d^2z = \left(\frac{d^2z}{dx^2}\right) dx^2 + 2 \frac{d^2z}{dx\,dy} dx\,dy + \left(\frac{d^2z}{dy^2}\right) dy^2.$$

$$(3)\ d^3z = \left(\frac{d^3z}{dx^3}\right) dx^3 + 3 \frac{d^3z}{dx^2\,dy} dx^2\,dy + 3 \frac{d^3z}{dx\,dy^2} dx\,dy^2 + \left(\frac{d^3z}{dy^3}\right) dy^3.$$

$$\ldots = \ldots\ldots\ldots\ldots$$

$$(n)\ d^nz = \left(\frac{d^nz}{dx^n}\right) dx^n + n \frac{d^nz}{dx^{n-1}\,dy} dx^{n-1}\,dy + \frac{n\,\overline{n-1}}{1\,.\,2} \frac{d^nz}{dx^{n-2}\,dy^2} dx^{n-2}\,dy^2 + \&c.$$

which are the first, second, third, and nth *total differentials* of the function $z = f(x, y)$.

The last formula (n) has been obtained by induction merely, and can only be assumed as true when n is an integer: but it may be demonstrated by the method of separation of symbols, as follows: since

$dz = \left(\frac{dz}{dx}\right) dx + \left(\frac{dz}{dy}\right) dy$; if n represent the index of operation

on both sides, we have

$$d^nz = \left(\left(\frac{d}{dx}\right) dx + \left(\frac{d}{dy}\right) dy\right)^n z$$

$$\therefore d^nz = \left(\frac{d^nz}{dx^n}\right) dx^n + n \frac{d^nz}{dx^{n-1}\,dy} dx^{n-1}\,dy + \frac{n\,\overline{n-1}}{1\,.\,2} \frac{d^nz}{dx^{n-2}\,dy^2} dx^{n-2}\,dy^2 + \frac{n\,\overline{n-1}\,\overline{n-2}}{1\,.\,2\,.\,3} \frac{d^nz}{dx^{n-3}\,dy^3} dx^{n-3}\,dy^3 + \&c.$$

This theorem is therefore true whether n be whole or fractional, positive or negative.

Ex. (1.) Let $z = \dfrac{ay}{\sqrt{x^2+y^2}}$

$$\left(\frac{dz}{dx}\right) dx = -\frac{axy\,dx}{(x^2+y^2)^{\frac{3}{2}}}$$

$$\left(\frac{dz}{dy}\right) dy = \frac{ax^2\,dy}{(x^2+y^2)^{\frac{3}{2}}}$$

But $dz = \left(\dfrac{dz}{dx}\right) dx + \left(\dfrac{dz}{dy}\right) dy$

$$\therefore dz = -\frac{axy\,dx - ax^2\,dy}{(x^2+y^2)^{\frac{3}{2}}}$$

Ex. (2.) Let $z = \log. \dfrac{x+\sqrt{x^2-y^2}}{x-\sqrt{x^2-y^2}} = \log. (x+\sqrt{x^2-y^2}) -$ $\log. (x-\sqrt{x^2-y^2})$

$$\left(\frac{dz}{dx}\right) = \frac{1+x(x^2-y^2)^{-\frac{1}{2}}}{x+\sqrt{x^2-y^2}} + \frac{x(x^2-y^2)^{-\frac{1}{2}}-1}{x-\sqrt{x^2-y^2}} = \frac{2}{\sqrt{x^2-y^2}}$$

$$\left(\frac{dz}{dy}\right) = -\frac{y(x^2-y^2)^{-\frac{1}{2}}}{x+\sqrt{x^2-y^2}} - \frac{y(x^2-y^2)^{-\frac{1}{2}}}{x-\sqrt{x^2-y^2}} = -\frac{2x}{y\sqrt{x^2-y^2}}$$

$$\therefore dz = \left(\frac{dz}{dx}\right) dx + \left(\frac{dz}{dy}\right) dy = \frac{2\,dx}{\sqrt{x^2-y^2}} - \frac{2x\,dy}{y\sqrt{x^2-y^2}} =$$

$$\frac{2y\,dx - 2x\,dy}{y\sqrt{x^2-y^2}}$$

Ex. (3.) Let $z = \tan. \dfrac{x}{y}$

$$\left(\frac{dz}{dx}\right) = \frac{1}{y} \sec.^2 \frac{x}{y}, \left(\frac{dz}{dy}\right) = -\frac{x}{y^2} \sec.^2 \frac{x}{y}$$

$$\therefore dz = \left(\frac{dz}{dx}\right) dx + \left(\frac{dz}{dy}\right) dy = \frac{y\,dx - x\,dy}{y^2} \sec.^2 \frac{x}{y}.$$

(82.) Let $u = f(x, y, z)$ it is required to find the development of $u' = f(x + h, y + k, z + l)$ where h, k, and l are any magnitudes, whole or fractional, positive or negative.

Let z remain constant, while x and y become $x + h$ and $y + k$ respectively, then $f(x + h, y + k, z) = u + \left(\frac{du}{dx}\right)h + \left(\frac{du}{dy}\right)k + \&c.$

Let z become $z + l$, then

$$u = u + \frac{du}{dz} l + \&c.$$

$$\frac{du}{dx} = \frac{du}{dx} + \frac{d^2u}{dz\,dx} l + \&c.$$

$$\frac{du}{dy} = \frac{du}{dy} + \frac{d^2u}{dz\,dy} l + \&c.$$

&c. = &c.

$$\therefore u' = f(x + h, y + k, z + l) = u + \frac{du}{dx} h + \frac{du}{dy} k + \frac{du}{dz} l + \&c.$$

In a similar manner may functions of four or more variables be developed.

Cor. Let h, k and l become dx, dy, and dz, then

$$du = \left(\frac{du}{dx}\right) dx + \left(\frac{du}{dy}\right) dy + \left(\frac{du}{dz}\right) dz.$$

Ex. (1.) Let $u = xyz$

$$\frac{du}{dx} = yz,\ \frac{du}{dy} = xz,\ \frac{du}{dz} = xy$$

$$\therefore\ du = \left(\frac{du}{dx}\right) dx + \left(\frac{du}{dy}\right) dy + \left(\frac{du}{dz}\right) dz =$$

$$yz\,dx + xz\,dy + xy\,dz.$$

Ex. (2.) Let $u = \dfrac{1}{\sqrt{(a-x)^2 + (b-y)^2 + (c-z)^2}}$ then

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = 0.$$

For $\dfrac{1}{u^2} = (a-x)^2 + (b-y)^2 + (c-z)^2$

$$\therefore\ \frac{du}{dx} = u^3(a-x)$$

$$\frac{d^2u}{dx^2} = 3u^2\frac{du}{dx}(a-x) - u^3 = 3u^5(a-x)^2 - u^3.$$

In a similar manner it appears that

$$\frac{d^2u}{dy^2} = 3u^5(b-y)^2 - u^3$$

and $\dfrac{d^2u}{dz^2} = 3u^5(c-z)^2 - u^3$

$$\therefore\ \frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = 3u^5\left((a-x)^2 + (b-y)^2 + (c-z)^2\right) - 3u^3$$

$$= 3u^3 - 3u^3 = 0.$$

(83.) Let $z = f(x, y) = 0$ be an implicit function of x and y, then $z = z + \left(\dfrac{dz}{dx}\right) h + \left(\dfrac{dz}{dy}\right) k + \&c.$ But $z = 0$ whatever the values

of x and y are. $\therefore z' = 0$, and $\therefore \left(\frac{dz}{dx}\right) h + \left(\frac{dz}{dy}\right) k + \&c. = 0$.

Let h and k become dx and dy, then $\left(\frac{dz}{dx}\right) dx + \left(\frac{dz}{dy}\right) dy = 0$;

that is, $\left(\frac{dz}{dx}\right) + \left(\frac{dz}{dy}\right) \frac{dy}{dx} = 0$.

(84.) Let v be any function of x and y, $f(v)$ any function of v, then

$$\frac{d}{dx}\left(f(v)\frac{dv}{dy}\right) = \frac{d}{dy}\left(f(v)\frac{dv}{dx}\right)$$

For let u be such a function of v that $\frac{du}{dv} = f(v)$, then $\frac{d}{dx}\left(f(v)\frac{dv}{dy}\right)$

$$= \frac{d}{dx}\frac{du}{dv}\frac{dv}{dy} = \frac{d}{dy}\left(\frac{du}{dv}\cdot\frac{dv}{dx}\right) \quad \therefore \frac{d}{dx}\left(f(v)\frac{dv}{dy}\right) = \frac{d}{dy}\left(f(v)\frac{dv}{dx}\right).$$

CHAPTER VIII.

DEVELOPMENT OF FUNCTIONS BY LAGRANGE'S THEOREM AND LAPLACE'S THEOREM.

(85.) Let $y = z + x\,\varphi(y)$, when x and z are independent of each other, and $u = f(y)$, then

$$u = f(z) + \varphi(z)\frac{d\,f(z)}{dz}\,\frac{x}{1} + \frac{d}{dz}\left((\varphi(z))^2\,\frac{d\,f(z)}{dz}\right)\frac{x^2}{1\,.\,2} +$$

$$\frac{d^2}{dz^2}\left((\varphi(z))^3\,\frac{d\,f(z)}{dz}\right)\frac{x^3}{1\,.\,2\,.\,3} + \ldots\ldots + \frac{d^{n-1}}{dz^{n-1}}\left((\varphi(z))^n\,\frac{d\,f(z)}{dz}\right)\frac{x^n}{1\,.\,2\,.\,3\ldots n}$$

$+$ &c.

For $y = z + x\,\varphi(y)$

$$\frac{dy}{dx} = \varphi(y) + x\,\frac{d\,\varphi(y)}{dy}\,\frac{dy}{dx} \quad \therefore \frac{dy}{dx} = \frac{\varphi(y)}{1 - x\,\frac{d\,\varphi(y)}{dy}} \quad (1)$$

$$\frac{dy}{dz} = 1 + x\,\frac{d\,\varphi(y)}{dy}\,\frac{dy}{dz} \quad \therefore \frac{dy}{dz} = \frac{1}{1 - x\,\frac{d\,\varphi(y)}{dy}} \quad \therefore \varphi(y)\,\frac{dy}{dz}$$

$$= \frac{\varphi(y)}{1 - x\,\frac{d\,\varphi(y)}{dy}} \quad \therefore \frac{dy}{dx} = \varphi(y)\,\frac{dy}{dz} \text{ by (1).}$$

But $u = f(y)$;

$$\therefore \frac{du}{dx} = \frac{d\,f(y)}{dy}\,\frac{dy}{dx} = \varphi(y)\,\frac{du}{dy}\,\frac{dy}{dz} = \varphi(y)\,\frac{du}{dz}$$

$$\frac{d^2u}{dx^2} = \frac{d}{dx}\left(\varphi(y)\frac{du}{dz}\right) = \frac{d}{dz}\left(\varphi(y)\frac{du}{dx}\right)(84) = \frac{d}{dz}\left(\overline{\varphi(y)}^2\frac{du}{dz}\right)$$

$$\frac{d^3u}{dx^3} = \frac{d^2}{dz\,dx}\left(\overline{\varphi(y)}^2\frac{du}{dz}\right) = \frac{d^2}{dz^2}\left(\overline{\varphi(y)}^2\frac{du}{dx}\right) = \frac{d^2}{dz^2}\left(\overline{\varphi(y)}^3\frac{du}{dz}\right)$$

$$\ldots = \ldots\ldots\ldots$$

$$\frac{d^nu}{dx^n} = \frac{d^{n-1}}{dz^{n-1}}\left(\overline{\varphi(y)}^n\frac{du}{dz}\right)$$

$$\frac{d^{n+1}u}{dx^{n+1}} = \frac{d^{n-1}}{dz^{n-1}}\frac{d}{dx}\left(\overline{\varphi(y)}^n\frac{du}{dz}\right) = \frac{d^n}{dz^n}\left(\overline{\varphi(y)}^n\frac{du}{dx}\right) = \frac{d^n}{dz^n}\left(\overline{\varphi(y)}^{n+1}\frac{du}{dz}\right)$$

Therefore, if the assumed value of $\frac{du}{dx}$ be true for any value of n, it is also true for $n+1$. But it is true for 1, 2, 3, 4, &c.; $\therefore$ it is true for n, and consequently universally true.

Let $x = 0$, then $y = z + x\varphi(y) = z$, $\varphi(y) = \varphi(z)$.

$u = f(y) = f(z)$, and $\frac{d^nu}{dx^n} = \frac{d^{n-1}}{dz^{n-1}}\left(\overline{\varphi(z)}^n\frac{df(z)}{dz}\right)$

But $u = (u) + \left(\frac{du}{dx}\right)\frac{x}{1} + \left(\frac{d^2u}{dx^2}\right)\frac{x^2}{1.2} + \left(\frac{d^3u}{dx^3}\right)\frac{x^3}{1.2.3} + \ldots$

$+ \left(\frac{d^nu}{dx^n}\right)\frac{x^n}{1.2\ldots n} +$ &c. $\therefore f(y) = f(z) + \varphi(z)\frac{df(z)}{dz}\frac{x}{1} +$
$\frac{d}{dz}\left(\overline{\varphi(z)}^2\frac{df(z)}{dz}\right)\frac{x^2}{1.2} + \frac{d^2}{dz^2}\left(\overline{\varphi(z)}^3\frac{df(z)}{dz}\right)\frac{x^3}{1.2.3} + \ldots +$
$\frac{d^{n-1}}{dz^{n-1}}\left(\overline{\varphi(z)}^n\frac{df(z)}{dz}\right)\frac{x^n}{1.2.3\ldots n} +$ &c., which is *Lagrange's Theorem*, who has expressed it in the following form,

$$f\,y = f\,z + \frac{x}{1} f'\,z\,\varphi z + \frac{x^2}{1\,.\,2}(f'\,z\,\varphi z^2)' + \frac{x^3}{1\,.\,2\,.\,3}(f'\,z\,\varphi z^3)''$$

$+$ &c.

Vide Theorie des Fonctions, page 149.

Example (1.) Let $1 - y + ay = 0$; find the value of $y^{\frac{1}{2}}$ by Lagrange's Theorem.

$$z = 1,\ x = a,\ \varphi(y) = y, \text{ and } f(z) = z^{\frac{1}{2}} = 1.$$

$$\varphi(z)\frac{d\,f(z)}{dz} = \frac{1}{2}z^{\frac{1}{2}} = \frac{1}{2},\ \frac{d}{dz}\left(\overline{\varphi(z)})^2\frac{d\,f(z)}{dz}\right) = \frac{1\,.\,3}{2^2}$$

$$\frac{d^2}{dz^2}\left(\overline{\varphi(z)})^3\frac{d\,f(z)}{dz}\right) = \frac{1\,.\,3\,.\,5}{2^3},\ \text{and}\ \frac{d^3}{dz^3}\left(\overline{\varphi(z)})^4\frac{d\,f(z)}{dz}\right) = \frac{1\,.\,3\,.\,5\,.\,7}{2^4}$$

$$\therefore y^{\frac{1}{2}} = 1 + \frac{1}{2}a + \frac{1\,.\,3}{2^3}a^2 + \frac{1\,.\,3\,.\,5}{2^4\,.\,3}a^3 + \frac{1\,.\,3\,.\,5\,.\,7}{2^5\,.\,3\,.\,4}a^4 + \&c.$$

$$= 1 + \frac{1}{2}a + \frac{3}{8}a^2 + \frac{5}{16}a^3 + \frac{35}{128}a^4 + \&c.$$

Ex. (2.) Let $1 - y + ay = 0$; find $\log.\ y$ by Lagrange's Theorem.

$$f(z) = \log.\ z$$

$$\varphi(z)\frac{d\,f(z)}{dz} = 1$$

$$\frac{d}{dz}\left(\overline{\varphi(z)})^2\frac{d\,f(z)}{dz}\right) = 1$$

$$\frac{d^2}{dz^2}\left(\overline{\varphi(z)})^3\frac{d\,f(z)}{dz}\right) = 1\,.\,2$$

$$\frac{d^3}{dz^3}\left(\varphi(z))^4\frac{d\,f(z)}{dz}\right) = 1\,.\,2\,.\,3$$

$$\frac{d^4}{dz^4}\left((\varphi(z))^5 \frac{d\,f(z)}{dz}\right) = 1 \,.\, 2 \,.\, 3 \,.\, 4$$

$$\&c. \qquad = \qquad \&c.$$

$$\therefore \log. y = a + \frac{a^2}{2} + \frac{a^3}{3} + \frac{a^4}{4} + \frac{a^5}{5} + \&c.$$

Ex. (3.) Let $\theta = u + e \sin u$; find u in terms of θ by Lagrange's Theorem.

$$u = \theta - e \sin. u$$

$$f(\theta) = \theta,\ \varphi(\theta) = \sin. \theta,\ x = -e$$

$$\varphi(\theta) \frac{d f \theta}{d \theta} = \sin. \theta$$

$$\frac{d}{d\theta}\left((\varphi(\theta))^2 \frac{d f(\theta)}{d\theta}\right) = 2 \sin. \theta \cos. \theta = \sin. 2\theta$$

$$\frac{d^2}{d\theta^2}\left((\varphi(\theta))^3 \frac{d f(\theta)}{d\theta}\right) = 6 \sin. \theta - 9 \sin.^3 \theta = \frac{3^2 \sin. 3\theta - 3 \sin. \theta}{2^2}$$

$$\&c. \quad \&c. \qquad = \qquad \&c. \quad \&c.$$

$$\therefore u = \theta - \frac{e}{1} \sin. \theta + \frac{e^2}{1\,.\,2} \sin. 2\theta - \frac{e^3}{1\,.\,2\,.\,3} \frac{3^2 \sin. 3\theta - 3 \sin. \theta}{2^2}$$

$$+ \&c.$$

Ex. (4.) Let $nt = (\theta - B) - 2e \sin. (\theta - B) + \frac{3}{4} e^2 \sin. 2(\theta - B) - \frac{e^3}{3}$ $\sin. 3(\theta - B)$ to find $\theta - B$ the true anomaly of a planet in terms of nt, the mean anomaly as far as e^3, e being a small fraction.

$$\text{Here } \theta - B = nt + e\left(2 \sin. (\theta - B) - \frac{3}{4} e \sin. 2(\theta - B) + \frac{e^2}{3} \sin. 3(\theta - B)\right)$$

$$y = z + e\left(2 \sin. y - \frac{3e}{4} \sin. 2y + \frac{e^2}{3} \sin. 3y\right)$$

$$\varphi(z) = 2 \sin. z - \frac{3e}{4} \sin. 2z + \frac{e^2}{3} \sin. 3z$$

$$\overline{\varphi(z))}^2 = (2 \sin. z - \frac{3e}{4} \sin. 2z)^2 = 2 - 2 \cos. 2z - \frac{3e}{2} \cos. z + \frac{3e}{2} \cos. 3z.$$

$$\therefore \frac{d}{dz} (\overline{(\varphi(z))^2}) = 4 \sin. 2z + \frac{3e}{2} \sin. z - \frac{9e}{2} \sin. 3z.$$

$$\overline{\varphi(z))}^3 = 8 \sin.^3 z = 6 \sin. z - 2 \sin. 3z.$$

$$\therefore \frac{d^2}{dz^2} (\overline{(\varphi(z))^3}) = -6 \sin. z + 18 \sin. 3z.$$

$$\therefore y = z + (2 \sin. z - \frac{3e}{4} \sin. 2z + \frac{e^2}{3} \sin. 3z) \frac{e}{1} +$$

$$+ (4 \sin. 2z + \frac{3e}{2} \sin. z - \frac{9e}{2} \sin. 3z) \frac{e^2}{2} + (18 \sin. 3z - 6 \sin. z) \frac{e^3}{6}$$

$$= z + \left(2e - \frac{e^3}{4}\right) \sin. z + \frac{5e^2}{4} \sin. 2z + \frac{13}{12} e^3 \sin. 3z + \&c.$$

That is $(\theta - B) = nt + \left(2e - \frac{e^3}{4}\right) \sin. nt + \frac{5e^2}{4} \sin. 2nt + \frac{13}{12} e^3 \sin. 3nt$ $+ \&c.$

Vide *Airy's Physical Astronomy*, page 11.

Ex. (5.) Let $1 - y + ay = 0$, then $y^{-m} = 1 - ma + \frac{m\,\overline{m-1}}{1 \,.\, 2} a^2 - \frac{m\,\overline{m-1}\,\overline{m-2}}{1 \,.\, 2 \,.\, 3} a^3 + \&c.$

Ex. (6.) Let $\theta = u + e \sin. u$, then $\sin. u = \sin. \theta - \frac{e}{1} \frac{\sin. 2\theta}{2} + \frac{e^2}{1 \,.\, 2 \,.\, 2^2} (3 \sin. 3\theta - \sin. \theta) - \&c.$

Ex. (7.) Let $xy^n - y + a = 0$, find the value of y^2 by Lagrange's Theorem.

Ex. (8.) Let $y = a + x \log. y$, find y in terms of x by Lagrange's Theorem.

(86.) Let $y = \psi(z + x\phi(y))$ where x and z are independent quantities it is required to expand $u = f(y)$ in a series of ascending powers of x.

Since $y = \psi(z + x\phi(y))$,

$$\frac{dy}{dx} = \psi'(z + x\phi(y))\left(\phi(y) + x\phi'(y)\frac{dy}{dx}\right),$$

$$\frac{dy}{dz} = \psi'(z + x\phi(y))\left(1 + x\phi'(y)\frac{dy}{dz}\right);$$

$$\therefore \frac{dy}{dx}(1 - x\phi'(y)\psi'(z + x\phi(y)) = \psi'(z + x\phi(y))\,\phi(y),$$

$$\text{and } \frac{dy}{dz}(1 - x\phi'(y)\psi'(z + x\phi(y)) = \psi'(z + x\phi(y));$$

$\therefore \dfrac{dy}{dx} = \phi(y)\dfrac{dy}{dz}$, and as this is of the same form as in the demonstration of Lagrange's Theorem, it follows that $\dfrac{d^n u}{dx^n} = \dfrac{d^{n-1}}{dz^{n-1}}\left(\overline{\phi(y)}^n\dfrac{du}{dz}\right)$.

Let $x = 0$ in the equation $y = \psi(z + x\phi(y))$, then $y = \psi(z)$, $\phi(y) = \phi(\psi(z))$, $u = f(y) = f(\psi(z))$ and $\dfrac{du}{dz} = \dfrac{df(\psi(z))}{dz}$ $\therefore$ we have as in (85)

$$f(y) = f(\psi(z)) + \phi(\psi(z))\frac{df(\psi(z))}{dz}\frac{x}{1} + \frac{d}{dz}\left(\overline{\phi(\psi(z))}^2\frac{df(\psi(z))}{dz}\right)\frac{x^2}{\lfloor 2} + \ldots + \frac{d^{n-1}}{dz^{n-1}}\left(\overline{\phi(\psi(z))}^n\frac{df(\psi(z))}{dz}\right)\frac{x^n}{\lfloor n} + \&c.,$$

$\lfloor n$ being $= 1 . 2 . 3 \ldots . n$, which is *Laplace's Theorem.*

EXAMPLE (1.) Let $y = \log. (z + x \sin. y)$, expand e^y in terms of x, by Laplace's Theorem.

$f(y) = e^y$, $\psi(z) = \log. z$, $\varphi(\psi(z)) = \sin. \log. z$.

$f(\psi(z)) = e^{\log. z} = z$, $\varphi(\psi(z)) \dfrac{df(\psi(z))}{dz} = \sin. \log. z$.

$$\frac{d}{dz}\left(\overline{\varphi(\psi(z))}^2 \frac{df(\psi(z))}{dz}\right) = 2\sin.(\log. z)\cos.(\log. z)\frac{1}{z} = \sin.(\log. z^2)\frac{1}{z}$$

$$\frac{d^2}{dz^2}\left(\overline{\varphi(\psi(z))}^3 \frac{df(\psi(z))}{dz}\right) = \frac{3\sin.(\log. z)}{z^2}(2 - 3\sin.^2(\log. z)$$

$$- \sin.(\log. z)\cos.(\log. z)) = \frac{3\sin.(\log. z)}{4z^2}(8 - 9\sin.(\log. z)$$

$$- 2\sin.(\log. z^2) + 3\sin.(\log. z^3)).$$

$$\therefore e^y = z + \sin.(\log. z)\frac{x}{1} + \frac{\sin.(\log. z^2)}{z}\frac{x^2}{\lfloor 2} +$$

$$+ \frac{3\sin.(\log. z)}{4z^2}(8 - 9\sin.(\log. z) - 2\sin.(\log. z^2) + 3\sin.(\log. z^3))\frac{x^3}{\lfloor 3}$$

$+$ &c.

Ex. (2.) Let $y = e^{z + x\cos. y}$, expand y in terms of x by Laplace's Theorem.

$f(y) = y$, and $\psi(z) = e^z$ $\therefore f(\psi(z)) = e^z$, $\varphi(z) = \cos. z$.

$$\varphi(\psi(z))\frac{df(\psi(z))}{dz} = e^z \cos. e^z.$$

$$\frac{d}{dz}\left(\overline{\varphi(\psi(z))}^2 \frac{df(\psi(z))}{dz}\right) = e^z\cos. e^z(\cos. e^z - 2\sin. e^z . e^z).$$

$$\frac{d^2}{dz^2}\left((\varphi_e(\psi(z)))^x \frac{df(\psi(z))}{dz}\right) = e^z \cos. e^z (\cos.^2 e^z - 9 e^z \cos. e^z \sin. e^z +$$

$$+ 9 e^{2z} \sin.^2 e^z - 3 e^{2z}).$$

$$\therefore y = e^z + e^z \cos. e^z \frac{x}{1} + e^z \cos. e^z (\cos. e^z - 2 \sin. e^z . e^z) \frac{x^2}{\underline{|2}}$$

$$+ e^z \cos. e^z (\cos.^2 e^z - 9 e^z \sin. e^z \cos. e^z + 9 e^{2z} \sin.^2 e^z - 3 e^{2z}) \frac{x^3}{\underline{|3}} + \&c.$$

CHAPTER IX.

CHANGE OF THE INDEPENDENT VARIABLE.

(87.) The differential coefficients $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, &c. have been obtained upon the hypothesis that x is the independent variable, it is required to find their values when y becomes the independent variable.

When x becomes $x + h$, let y become $y + k$, then, since $y = f(x)$,

$$k = \frac{dy}{dx} h + \frac{d^2y}{dx^2} \frac{h^2}{1 \cdot 2} + \frac{d^3y}{dx^3} \frac{h^3}{1 \cdot 2 \cdot 3} + \&c. \quad (1)$$

But since $y = f(x)$, $x = f^{-1}(y)$

$$\therefore h = \frac{dx}{dy} k + \frac{d^2x}{dy^2} \frac{k^2}{1 \cdot 2} + \frac{d^3x}{dy^3} \frac{k^3}{1 \cdot 2 \cdot 3} + \&c.$$

Substituting this value of h in (1), we have

$$k = \frac{dy}{dx} \left(\frac{dx}{dy} k + \frac{d^2x}{dy^2} \frac{k^2}{1 \cdot 2} + \frac{d^3x}{dy^3} \frac{k^3}{1 \cdot 2 \cdot 3} + \&c. \right)$$

$$+ \frac{1}{1 \cdot 2} \frac{d^2y}{dx^2} \left(\frac{dx^2}{dy^2} k^2 + \frac{dx}{dy} \frac{d^2x}{dy^2} k^3 + \&c. \right)$$

$$+ \frac{1}{1 \cdot 2 \cdot 3} \frac{d^3y}{dx^3} \left(\frac{dx^3}{dy^3} k^3 + \&c. \right)$$

$$+ \quad \&c.$$

$$= \frac{dy}{dx}\frac{dx}{dy}k + \left(\frac{dy}{dx}\frac{d^2x}{dy^2} + \frac{d^2y}{dx^2}\frac{dx^2}{dy^2}\right)\frac{k^2}{1.2} +$$

$$+ \left(\frac{d^3x}{dy^3}\frac{dy}{dx} + 3\frac{dx}{dy}\frac{d^2y}{dx^2}\frac{d^2x}{dy^2} + \frac{d^3y}{dx^3}\frac{dx^3}{dy^3}\right)\frac{k^3}{1.2.3} + \&c.$$

Therefore equating the coefficients of like powers of k,

$$1 = \frac{dy}{dx}\frac{dx}{dy} \therefore \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

$$\frac{dy}{dx}\frac{d^2x}{dy^2} + \frac{d^2y}{dx^2}\frac{dx^2}{dy^2} = 0 \therefore \frac{d^2y}{dx^2} = \frac{-\frac{d^2x}{dy^2}}{\frac{dx^3}{dy^3}}.$$

$$\frac{d^3x}{dy^3}\frac{dy}{dx} + 3\frac{dx}{dy}\frac{d^2y}{dx^2}\frac{d^2x}{dy^2} + \frac{d^3y}{dx^3}\frac{dx^3}{dy^3} = 0;$$

$$\therefore \frac{d^3x}{dy^3}\frac{dx}{dy} - 3\left(\frac{d^2x}{dy^2}\right)^2 + \frac{d^3y}{dx^3}\frac{dx^5}{dy^5} = 0;$$

$$\therefore \frac{d^3y}{dx^3} = \frac{3\left(\frac{d^2x}{dy^2}\right)^2 - \frac{dx}{dy}\frac{d^3x}{dy^3}}{\frac{dx^5}{dy^5}}.$$

These results may be deduced from Cor. of (23), for

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{\frac{dx}{dy}}\right) = \frac{1}{\frac{dx}{dy}}\frac{d}{dy}\left(\frac{1}{\frac{dx}{dy}}\right) = \frac{-\frac{d^2x}{dy^2}}{\frac{dx^3}{dy^3}}.$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{-\frac{d^2x}{dy^2}}{\frac{dx^3}{dy^3}}\right) = \frac{1}{\frac{dx}{dy}}\frac{d}{dy}\left(\frac{-\frac{d^2x}{dy^2}}{\frac{dx^3}{dy^3}}\right) =$$

$$\frac{1}{\frac{dx}{dy}}\;\frac{3\,\frac{dx^2}{dy^2}\left(\frac{d^2x}{dy^2}\right)^2 - \frac{dx^3}{dy^3}\,\frac{d^3x}{dy^3}}{\frac{dx^6}{dy^6}} = \frac{3\left(\frac{d^2x}{dy^2}\right)^2 - \frac{dx}{dy}\,\frac{d^3x}{dy^3}}{\frac{dx^5}{dy^5}}.$$

EXAMPLE (1.) Change $\frac{dy}{dx} - \frac{dy^3}{dx^3} = x\,\frac{d^2y}{dx^2}$ into a formula where y is the independent variable.

$$\frac{1}{\frac{dx}{dy}} - \frac{1}{\frac{dx^3}{dy^3}} = \frac{-\,x\,\frac{d^2x}{dy^2}}{\frac{dx^3}{dy^3}},$$

$$\frac{dx^2}{dy^2} - 1 = -\,x\,\frac{d^2x}{dy^2};$$

$$\therefore\; x\,\frac{d^2x}{dy^2} + \frac{dx^2}{dy^2} = 1.$$

Ex. (2.) Change the expression for the radius of curvature $\rho = \frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{-\,\frac{d^2y}{dx^2}}$ into one where y is the independent variable.

$$\rho = \frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{-\,\frac{d^2y}{dx^2}} = \frac{\left(1 + \frac{1}{\frac{dx^2}{dy^2}}\right)^{\frac{3}{2}}}{\frac{\frac{d^2x}{dy^2}}{\frac{dx^3}{dy^3}}} = \frac{\left(1 + \frac{dx^2}{dy^2}\right)^{\frac{3}{2}}}{\frac{d^2x}{dy^2}}.$$

Ex. (3.) Change the formula $\frac{d^2y}{dx^2} - x\,\frac{dy^2}{dx^2} + e^y\,\frac{dy^3}{dx^3} = 0$, into one where y is the independent variable.

$$\frac{d^2y}{dx^2} - x\frac{dy^2}{dx^2} + e^y\frac{dy^3}{dx^3} = -\frac{\dfrac{d^2x}{dy^2}}{\dfrac{dx^3}{dy^3}} - \frac{x}{\dfrac{dx^2}{dy^2}} + \frac{e^y}{\dfrac{dx^3}{dy^3}} = 0;$$

$$\therefore \frac{d^2x}{dy^2} + x\frac{dx}{dy} - e^y = 0.$$

(88.) Let x and y be functions of a third quantity θ, it is required to express $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, &c., in terms of $\frac{dx}{d\theta}$, $\frac{dy}{d\theta}$, $\frac{d^2x}{d\theta^2}$, $\frac{d^2y}{d\theta^2}$, &c.

Let θ become $\theta + m$; then x and y will become $x + h$ and $y + k$.

$$\therefore k = \frac{dy}{d\theta}m + \frac{d^2y}{d\theta^2}\frac{m^2}{1.2} + \frac{d^3y}{d\theta^3}\frac{m^3}{1.2.3} + \&c. \quad . \quad (1)$$

$$\text{And } h = \frac{dx}{d\theta}m + \frac{d^2x}{d\theta^2}\frac{m^2}{1.2} + \frac{d^3x}{d\theta^3}\frac{m^3}{1.2.3} + \&c. \quad . \quad (2)$$

Again, x and y are functions of the same quantity θ, they are therefore functions of each other; therefore when x becomes $x + h$, y becomes $y + k$.

$$\therefore k = \frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\frac{h^3}{1.2.3} + \&c.$$

Substitute for h its value in (2), we have

$$k = \frac{dy}{dx}\left(\frac{dx}{d\theta}m + \frac{d^2x}{d\theta^2}\frac{m^2}{1.2} + \frac{d^3x}{d\theta^3}\frac{m^3}{1.2.3} + \&c.\right)$$

$$+ \frac{1}{2}\frac{d^2y}{dx^2}\left(\frac{dx^2}{d\theta^2}m^2 + \frac{dx}{d\theta}\frac{d^2x}{d\theta^2}m^3 + \&c.\right)$$

$$+ \frac{1}{6}\frac{d^3y}{dx^3}\left(\frac{dx^3}{d\theta^3}m^3 + \&c.\right)$$

$$+ \&c.$$

$$= \frac{dy}{dx}\frac{dx}{d\theta}m + \left(\frac{dy}{dx}\frac{d^2x}{d\theta^2} + \frac{d^2y}{dx^2}\frac{dx^2}{d\theta^2}\right)\frac{m^2}{1\,.\,2} +$$

$$+ \left(\frac{dy}{dx}\frac{d^3x}{d\theta^3} + 3\frac{d^2y}{dx^2}\frac{dx}{d\theta}\frac{d^2x}{d\theta^2} + \frac{d^3y}{dx^3}\frac{dx^3}{d\theta^3}\right)\frac{m^3}{1\,.\,2\,.\,3} + \text{\&c.}$$

Equating the coefficients of like powers of m we have

$$\frac{dy}{d\theta} = \frac{dy}{dx}\frac{dx}{d\theta} \quad \therefore \frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}},$$

$$\frac{d^2y}{d\theta^2} = \frac{dy}{dx}\frac{d^2x}{d\theta^2} + \frac{d^2y}{dx^2}\frac{dx^2}{d\theta^2} \quad \therefore \frac{d^2y}{dx^2} = \frac{\dfrac{d^2y}{d\theta^2} - \dfrac{dy}{dx}\dfrac{d^2x}{d\theta^2}}{\dfrac{dx^2}{d\theta^2}}$$

$$= \frac{\dfrac{dx}{d\theta}\dfrac{d^2y}{d\theta^2} - \dfrac{dy}{d\theta}\dfrac{d^2x}{d\theta^2}}{\dfrac{dx^3}{d\theta^3}}.$$

In like manner it appears that

$$\frac{d^3y}{dx^3} = \frac{\dfrac{dx^2}{d\theta^2}\dfrac{d^3y}{d\theta^3} - 3\dfrac{dx}{d\theta}\dfrac{d^2x}{d\theta^2}\dfrac{d^2y}{d\theta^2} + 3\dfrac{dy}{d\theta}\left(\dfrac{d^2x}{d\theta^2}\right)^2 - \dfrac{dx}{d\theta}\dfrac{dy}{d\theta}\dfrac{d^3x}{d\theta^3}}{\dfrac{dx^5}{d\theta^5}}.$$

These results might have been deduced from (23) for

$$\frac{dy}{d\theta} = \frac{dy}{dx}\frac{dx}{d\theta} \quad \frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}},$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}}\right) = \frac{1}{\dfrac{dx}{d\theta}}\frac{d}{d\theta}\left(\frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}}\right) = \frac{\dfrac{dx}{d\theta}\dfrac{d^2y}{d\theta^2} - \dfrac{dy}{d\theta}\dfrac{d^2x}{d\theta^2}}{\dfrac{dx^3}{d\theta^3}},$$

and so on

EXAMPLE (1.) Transform $\dfrac{\dfrac{d^2y}{dx^2}}{\left(1+\left(\dfrac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}$ into a function where s is the independent variable, having given $\dfrac{ds^2}{dx^2} = 1 + \dfrac{dy^2}{dx^2}$.

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d^2y}{ds^2}\,\dfrac{dx}{ds} - \dfrac{d^2x}{ds^2}\,\dfrac{dy}{ds}}{\dfrac{dx^3}{ds^3}} \text{ and } 1 + \frac{dy^2}{dx^2} = 1 + \frac{\dfrac{dy^2}{ds}}{\dfrac{dx^2}{ds^2}} = \frac{\dfrac{dx^2+dy^2}{ds^2}}{\dfrac{dx^2}{ds^2}}$$

$$= \frac{1}{\dfrac{dx^2}{ds^2}} \therefore \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}} = \frac{1}{\dfrac{dx^3}{ds^3}} \therefore \frac{\dfrac{d^2y}{dx^2}}{\left(1 + \dfrac{dy^2}{dx^2}\right)^{\frac{3}{2}}} = \frac{d^2y}{ds^2}\,\frac{dx}{ds} - \frac{d^2x}{ds^2}\,\frac{dy}{ds}.$$

Ex. (2.) Transform $p = \dfrac{x\dfrac{dy}{dx} - y}{\left(1+\dfrac{dy^2}{dx^2}\right)^{\frac{1}{2}}}$ into a function of r and θ, having given $x = r \cos.\ \theta$, and $y = r \sin.\ \theta$.

Considering r a function of θ, we have by differentiation

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos.\ \theta - r\sin.\ \theta, \text{ and } \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin.\ \theta + r\cos.\ \theta;$$

$$\therefore x\frac{dy}{dx} = \frac{\dfrac{dr}{d\theta}\,r\sin.\ \theta\cos.\ \theta + r^2\cos.^2\theta}{\dfrac{dr}{d\theta}\cos.\ \theta - r\sin.\ \theta} \therefore x\frac{dy}{dx} - y = \frac{r^2}{\dfrac{dr}{d\theta}\cos.\ \theta - r\sin.\ \theta}.$$

$$\text{Again } \frac{dy}{dx} = \frac{\dfrac{dr}{d\theta}\sin.\ \theta + r\cos.\ \theta}{\dfrac{dr}{d\theta}\cos.\ \theta - r\sin.\ \theta} \therefore \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} = \frac{\left(\dfrac{dr^2}{d\theta^2} + r^2\right)^{\frac{1}{2}}}{\dfrac{dr}{d\theta}\cos.\ \theta - r\sin.\ \theta}$$

$$\therefore \frac{\frac{x\,dy}{dx} - y}{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}}} = \frac{r^2}{\left(r^2 + \frac{dr^2}{d\theta^2}\right)^{\frac{1}{2}}} = p.$$

Ex. (3.) Transform $\rho = \dfrac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}}$ into a function where θ is the independent variable, having given $x = r \cos. \theta$ and $y = r \sin. \theta$.

We have $\dfrac{dx}{d\theta} = \dfrac{dr}{d\theta} \cos. \theta - r \sin \theta$, $\dfrac{dy}{d\theta} = \dfrac{dr}{d\theta} \sin. \theta + r \cos. \theta$;

$$\therefore \frac{dy}{dx} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos. \theta}{\frac{dr}{d\theta} \cos. \theta - r \sin. \theta} \quad \therefore \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}} = \frac{\left(r^2 + \frac{dr^2}{d\theta^2}\right)^{\frac{3}{2}}}{\left(\frac{dr}{d\theta} \cos. \theta - r \sin. \theta\right)^3}.$$

$$\frac{d^2x}{d\theta^2} = - r \cos. \theta - 2 \sin. \theta \frac{dr}{d\theta} + \cos. \theta \frac{d^2r}{d\theta^2};$$

$$\frac{d^2y}{d\theta^2} = - r \sin. \theta + 2 \cos. \theta \frac{dr}{d\theta} + \sin. \theta \frac{d^2r}{d\theta^2},$$

$$\therefore \frac{\frac{dy}{d\theta} \frac{d^2x}{d\theta^2} - \frac{dx}{d\theta} \frac{d^2y}{d\theta^2}}{\frac{dx^3}{d\theta^3}} = - \frac{r^2 + 2 \frac{dr^2}{d\theta^2} - r \frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta} \cos. \theta - r \sin \theta\right)^3}.$$

$$\therefore \rho = - \frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}} = - \frac{\left(r^2 + \frac{dr^2}{d\theta^2}\right)^{\frac{3}{2}}}{r^2 + 2 \frac{dr^2}{d\theta^2} - r \frac{d^2r}{d\theta^2}}.$$

(89.) Let $u = f(x, y)$, $x = \varphi(r, \theta)$, and $y = \psi(r, \theta)$, it is required to express $\dfrac{du}{dx}, \dfrac{du}{dy}$ in terms of the variables r and θ.

$$\frac{du}{dr} = \frac{du}{dx}\frac{dx}{dr} + \frac{du}{dy}\frac{dy}{dr}, \qquad (1)$$

$$\frac{du}{d\theta} = \frac{du}{dx}\frac{dx}{d\theta} + \frac{du}{dy}\frac{dy}{d\theta}. \qquad (2)$$

Multiply (1) by $\frac{dy}{d\theta}$ and (2) by $\frac{dy}{dr}$, and subtract (2) from (1), and we have

$$\frac{du}{dx}\left(\frac{dx}{dr}\frac{dy}{d\theta} - \frac{dy}{dr}\frac{dx}{d\theta}\right) = \frac{du}{dr}\frac{dy}{d\theta} - \frac{du}{d\theta}\frac{dy}{dr},$$

$$\therefore \frac{du}{dx} = \frac{\frac{du}{dr}\frac{dy}{d\theta} - \frac{du}{d\theta}\frac{dy}{dr}}{\frac{dx}{dr}\frac{dy}{d\theta} - \frac{dy}{dr}\frac{dx}{d\theta}}.$$

Again, multiply (1) by $\frac{dx}{d\theta}$ and (2) by $\frac{dx}{dr}$, and subtract (2) from (1), and we have

$$-\frac{du}{dy}\left(\frac{dx}{dr}\frac{dy}{d\theta} - \frac{dx}{d\theta}\frac{dy}{dr}\right) = \frac{du}{dr}\frac{dx}{d\theta} - \frac{du}{d\theta}\frac{dx}{dr},$$

$$\therefore \frac{du}{dy} = -\frac{\frac{du}{dr}\frac{dx}{d\theta} - \frac{du}{d\theta}\frac{dx}{dr}}{\frac{dx}{dr}\frac{dy}{d\theta} - \frac{dx}{d\theta}\frac{dy}{dr}}.$$

EXAMPLE (1.) Transpose $x\frac{dR}{dx} + y\frac{dR}{dy}$ having given $x = r\cos.\theta$, $y = r\sin.\theta$, and therefore $x^2 + y^2 = r^2$, $\tan.\theta = \frac{y}{x}$, $\frac{dx}{dr} = \cos.\theta$, $\frac{dy}{dr} = \sin.\theta$, $\frac{dx}{d\theta} = -r\sin.\theta$, and $\frac{dy}{d\theta} = r\cos.\theta$.

But $\frac{d\,R}{dx} = \dfrac{\frac{d\,R}{dr}\frac{dy}{d\theta} - \frac{d\,R}{d\theta}\frac{dy}{dr}}{\frac{dx}{dr}\frac{dy}{d\theta} - \frac{dy}{dr}\frac{dx}{d\theta}} = \dfrac{\frac{d\,R}{dr}r\cos.\theta - \frac{d\,R}{d\theta}\sin.\theta}{r\cos.^2\theta + r\sin.^2\theta} =$

$\frac{d\,R}{dr}\cos.\theta - \frac{d\,R}{d\theta}\frac{\sin.\theta}{r}$. Again $\frac{d\,R}{dy} = -\dfrac{\frac{d\,R}{dr}\frac{dx}{d\theta} - \frac{d\,R}{d\theta}\frac{dx}{dr}}{\frac{dx}{dr}\frac{dy}{d\theta} - \frac{dy}{dr}\frac{dx}{d\theta}} =$

$$= \frac{\frac{d\,R}{dr}r\sin.\theta + \frac{d\,R}{d\theta}\cos.\theta}{r\cos.^2\theta + r\sin.^2\theta} = \frac{d\,R}{dr}\sin.\theta + \frac{d\,R}{d\theta}\frac{\cos.\theta}{r}.$$

$$\therefore \frac{x\,d\,R}{dx} + \frac{y\,d\,R}{dy} = r\frac{d\,R}{dr}.$$

Ex. (2.) Transform $\frac{d^2\varphi}{dx^2} + \frac{d^2\varphi}{dy^2} + \frac{d^2\varphi}{dz^2} = 0$ when $x^2 + y^2 + z^2 = r^2$.

We have $\frac{dr}{dx} = \frac{x}{r}$, $\frac{dr}{dy} = \frac{y}{r}$ and $\frac{dr}{dz} = \frac{z}{r}$.

Also $\frac{d\varphi}{dx} = \frac{d\varphi}{dr}\frac{dr}{dx} = \frac{d\varphi}{dr}\frac{x}{r}$ $\therefore \frac{d^2\varphi}{dx^2} = \frac{d^2\varphi}{dr^2}\frac{dr}{dx}\frac{x}{r} + \frac{d\varphi}{dr}\frac{1}{r}$

$- \frac{d\varphi}{dr}\frac{x}{r^2}\frac{dr}{dx} = \frac{d^2\varphi}{dr^2}\frac{x^2}{r^2} + \frac{d\varphi}{dr}\left(\frac{1}{r} - \frac{x^2}{r^3}\right)$. In a similar manner it appears that $\frac{d^2\varphi}{dy^2} = \frac{d^2\varphi}{dr^2}\frac{y^2}{r^2} + \frac{d\varphi}{dr}\left(\frac{1}{r} - \frac{y^2}{r^3}\right)$ and $\frac{d^2\varphi}{dz^2} = \frac{d^2\varphi}{dr^2}\frac{z^2}{r^2} + \frac{d\varphi}{dr}\left(\frac{1}{r} - \frac{z^2}{r^3}\right)$.

$$\therefore \frac{d^2\varphi}{dx^2} + \frac{d^2\varphi}{dy^2} + \frac{d^2\varphi}{dz^2} = \frac{d^2\varphi}{dr^2} + \frac{2}{r}\frac{d\varphi}{dr} = 0.$$

Ex. (3.) Transform the double integral $\iint e^{x^2+y^2}\,dx\,dy$ into one where r and θ are the independent variables, having given $x = r\cos.\theta$, and $y = r\sin.\theta$.

Since $x = r\cos.\theta$ and $y = r\sin.\theta \therefore x^2 + y^2 = r^2$, and $\frac{dx}{dr} = \cos.\theta$,

$\frac{dx}{d\theta} = -r\sin.\theta$, $\frac{dy}{dr} = \sin.\theta$, $\frac{dy}{d\theta} = r\cos.\theta$.

But $dx\,dy = \left(\frac{dx}{d\theta}\frac{dy}{dr} - \frac{dx}{dr}\frac{dy}{d\theta}\right) dr\,d\theta = -(r\sin.^2\theta + r\cos.^2\theta)\,dr\,d\theta$

$= -r\,dr\,d\theta$, and consequently $\iint e^{x^2+y^2}\,dx\,dy = -\iint e^{r^2}\,r\,dr\,d\theta$.

Examples for Practice.

(1.) If $y\frac{d^2y}{dx^2} + 2\frac{dy^2}{dx^2} = y$, where x is the independent variable, then will $y\frac{d^2x}{dy^2} + y\frac{dx^3}{dy^3} = 2\frac{dx}{dy}$, where y is the independent variable.

(2.) If $x\frac{d^2y}{dx^2} - \frac{dy}{dx} + x\frac{dy^2}{dx^2} = 0$, where x is the independent variable, then will $x\frac{d^2x}{dy^2} + \frac{dx^2}{dy^2} = x\frac{dx}{dy}$, where y is the independent variable.

(3.) If $\frac{dz}{dy} + \frac{z}{(1+y^2)^{\frac{1}{2}}} = a$, where y is the independent variable, then will $\frac{dz}{dx} + z = \frac{a}{2}(e^x + e^{-x})$, where x is the independent variable, and $e^x = y + (1+y^2)^{\frac{1}{2}}$.

(4.) If $\frac{d^2z}{dx^2} + n^2z = 0$, where x is the independent variable, then will $(1-y^2)\frac{d^2z}{dy^2} - y\frac{dz}{dy} + x^2z = 0$, where y is the independent variable, and $y = \cos.x$.

(5.) If $x \frac{dz}{dy} - y \frac{dz}{dx} = 0$, then $\frac{dz}{d\theta} = 0$, when $x = r \cos. \theta$ and $y = r \sin. \theta$.

(6.) If $\frac{d^2 y}{dx^2} - \frac{x}{1-x} \frac{dy}{dx} + \frac{y}{1-x^2} = 0$, then will $\frac{d^2 y}{d\theta^2} + y = 0$, when $\theta = \cos.^{-1} x$.

(7.) If $\frac{d^2 \varphi}{dx^2} + \frac{d^2 \varphi}{dy^2} = 0$, then $\frac{d^2 \varphi}{dr^2} + \frac{1}{r} \frac{d\varphi}{dr} = 0$, when $x^2 + y^2 = r^2$.

(8.) Transform $\int\int x^{m-1} y^{n-1} dx\, dy$ into a double integral, where u and v are the independent variables, having given $x + y = u$, and $y = uv$.—Ans. $\int\int x^{m-1} y^{n-1} dx\, dy = \int\int u^{m+n-1} (1 - v)^{m-1} v^{n-1} du\, dv$.

CHAPTER X.

MAXIMA AND MINIMA OF FUNCTIONS OF TWO OR MORE VARIABLES.

(70.) Let $u = f(x, y)$, and $u' = f(x + h, y + k)$, then

$$u' = u + \frac{du}{dx} h + \frac{d^2 u}{dx^2} \frac{h^2}{1 . 2} + \&c.$$

$$+ \frac{du}{dy} k + \frac{d^2 u}{dy\, dx} k\, h + \&c.$$

$$+ \frac{d^2 u}{dy^2} \frac{k^2}{1 . 2} + \&c.$$

Let $k = m\, h$, then

$$u' = u + h \left(\frac{du}{dx} + \frac{du}{dy} m\right) + \frac{h^2}{1 . 2} \left(\frac{d^2 u}{dx^2} + 2 \frac{d^2 u}{dx\, dy} m + \frac{d^2 u}{dy^2} m^2\right)$$

$+$ terms in h^3, h^4, &c.

Now it is necessary to a maximum or minimum that $u' - u$ may always have the same sign, whatever values be given to h and k. But h may be taken so small that $h \left(\frac{du}{dx} + \frac{du}{dy} m\right)$ will be greater than the sum of all the terms that follow it (63); $\therefore$ in order that $u' - u$ may always have the same sign, $h \left(\frac{du}{dx} + \frac{du}{dy} m\right)$ must be $= 0$.

$\therefore \frac{du}{dx} + \frac{du}{dy} m = 0$. But k is arbitrary $\therefore m$ is also arbitrary; consequently this equation must hold whatever be the value of m:

$\therefore \frac{du}{dx} = 0$, and $\frac{du}{dy} = 0$.

(91.) $f(x, y)$ will be a maximum or minimum according as $u' - u$ is negative or positive; we shall therefore proceed to enquire when this is the case.

Since $h \left(\frac{du}{dx} + \frac{du}{dy} m\right) = 0$, we have $u' - u = \frac{h^2}{2} \left(\frac{d^2u}{dx^2} + 2 \frac{d^2u}{dx\,dy} m + \frac{d^2u}{dy^2} m^2\right)$ + terms in h^3, h^4, &c. But h may be taken so small that $\frac{h^2}{1.2} \left(\frac{d^2u}{dx^2} + 2 \frac{d^2u}{dx\,dy} m + \frac{d^2u}{dy^2} m^2\right)$ may be greater than the sum of all the terms that follow it; and as $\frac{h^2}{2}$ is always positive, the sign of $u' - u$ will be the same as that of $\frac{d^2u}{dx^2} + 2 \frac{d^2u}{dx\,dy} m + \frac{d^2u}{dy^2} m^2$; $\therefore$ in order to a maximum or minimum, this sign must be incapable of changing, whatever be the value of m. Let $\frac{d^2u}{dx^2} = A$, $\frac{d^2u}{dx\,dy} = B$, and $\frac{d^2u}{dy^2} = C$, then $\frac{d^2u}{dx^2} + 2 \frac{d^2u}{dx\,dy} m + \frac{d^2u}{dy^2} m^2 = A + 2Bm + Cm^2 = A\left(1 + \frac{2B}{A} m + \frac{C}{A} m^2\right) = A \left(\left(1 + \frac{B}{A} m\right)^2 + \frac{AC - B^2}{A^2} m^2\right)$. $\therefore$ the sign of the quantity within the brackets will be positive whenever A and C have the same sign, and A C is not less

than B^2. Consequently, in order that $f(x, y)$ may be a maximum or minimum, $\frac{d^2u}{dx^2}$ and $\frac{d^2u}{dy^2}$ must have the same sign, and $\frac{d^2u}{dx^2} \cdot \frac{d^2u}{dy^2}$ must not be less than $\left(\frac{d^2u}{dx\,dy}\right)^2$; and it will be a maximum or minimum, according as the common sign of $\frac{d^2u}{dx^2}$ and $\frac{d^2u}{dy^2}$ is negative or positive.

If the values of x, y, which cause the first differential coefficient to vanish, cause the second to vanish also, $f(x, y)$ cannot be a maximum or minimum unless the third vanish, and the fourth be incapable of changing its sign.

EXAMPLE (1.) Let $u = x^4 + y^4 - 4axy^2$.

$$\frac{du}{dx} = 4x^3 - 4ay^2 = 0 \therefore y^2 = \frac{x^3}{a},$$

$$\frac{du}{dy} = 4y^3 - 8axy = 0 \therefore y^2 = 2ax;$$

$$\therefore \frac{x^3}{a} = 2ax, \text{ and } x = a\sqrt{2} \therefore y = 2^{\frac{3}{4}}a.$$

Also, $\frac{d^2u}{dx^2} = 12x^2 = 24a^2$, $\frac{d^2u}{dy^2} = 12y^2 - 8ax = 16a^2\sqrt{2}$.

$$\frac{d^2u}{dy\,dx} = -8ay = -8 \times 2^{\frac{3}{4}}a^2. \text{ But } 24a^2 \times 16a^2\sqrt{2} >$$

$(-8 \times 2^{\frac{3}{4}}a^2)^2$. $\therefore u = -4a^4$, which is a minimum.

Ex. (2.) Let $u = x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

$$\frac{du}{dx} = x^3 - x + y = 0 \therefore y = x - x^3,$$

$$\frac{du}{dy} = y^3 + x - y = 0 \therefore y^3 = y - x = -x^3 \therefore y = -x;$$

$$\therefore x^3 - x - x = 0; \;\therefore x = 0 \text{ or } x = \pm\sqrt{2}; \quad \frac{d^2u}{dx^2} = -1 \text{ or } 5;$$

$$y = 0 \text{ or } y = \mp\sqrt{2}, \quad \frac{d^2u}{dy\,dx} = 1, \text{ and } \frac{d^2u}{dy^2} = -1 \text{ or } 5;$$

$$\therefore x = 0,\ y = 0, \text{ give } u = 0 \text{ a maximum,}$$

$$x = \pm\sqrt{2},\ y = \mp\sqrt{2}, \text{ give } u = -8 \text{ a minimum.}$$

Ex. (3.) Find the maximum value of $u = al + bm + cn$, l, m, and n being variable and subject to the condition $l^2 + m^2 + n^2 = 1$,

$$n = \sqrt{1 - l^2 - m^2}, \text{ and } u = al + bm + c\sqrt{1 - l^2 - m^2},$$

$$\frac{du}{dl} = a - \frac{cl}{\sqrt{1 - l^2 - m^2}} = 0; \;\therefore a\sqrt{1 - l^2 - m^2} - cl = 0, \qquad (1)$$

$$\frac{du}{dm} = b - \frac{cm}{\sqrt{1 - l^2 - m^2}} = 0; \;\therefore b\sqrt{1 - l^2 - m^2} - cm = 0. \qquad (2)$$

Multiply (1) by b and (2) by a, and subtract, and we have

$$acm = bcl \;\therefore \frac{a}{l} = \frac{b}{m} = \frac{c}{n} \;\therefore m = \frac{b}{a}\,l \text{ and } n = \frac{c}{a}\,l \;\therefore l^2 + \frac{b^2}{a^2}\,l^2 + \frac{c^2}{a^2}\,l^2 = 1, \text{ that is } l = \frac{a}{\sqrt{a^2 + b^2 + c^2}},\ m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \text{ and}$$

$$n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}. \quad \therefore u = \sqrt{a^2 + b^2 + c^2}.$$

This is the solution of the problem, "To find the position of the plane on which the sum of the projections of any number of planes is a maximum," l, m, and n being the cosines of the angles which the plane of projection makes with the co-ordinate planes. (Vide *Gregory's Diff. Calculus*, page 114.)

Ex. (4.) Find the maximum or minimum value of $u = xyz$ subject to the condition expressed by the equation $a^x b^y c^z = k$

$\log. u = \log. x + \log. y + \log. z$, and $\log. k = x \log. a + y \log. b + z \log. c$,

$$\therefore z = \frac{\log. k - x \log. a - y \log. b}{\log. c}.$$

$$\therefore \log. u = \log. x + \log. y + \log. \frac{\log. k - x \log. a - y \log. b}{\log. c}.$$

$$\frac{du}{u\,dx} = \frac{1}{x} - \frac{\log. a}{\log. k - x \log. a - y \log b} = 0; \therefore \log. k - 2x \log. a - y$$

$\log. b = 0.$ (1.)

In a similar manner it appears that $\log. k - x \log. a - 2y \log. b = 0$. (2.)

Multiply (1) by 2, and subtract (2) from the product, and we have

$\log. k - 3x \log. a = 0; \therefore x = \dfrac{\log. k}{3 \log. a}$. In like manner it appears

that $y = \dfrac{\log. k}{3 \log. b}$, and $z = \dfrac{\log. k}{3 \log. c}$.

$\therefore u = \dfrac{\log.^3 k}{27 \log. a \log. b \log. c}$, which, by the usual criterion, is found

to be a maximum.

Ex. (5.) To inscribe the greatest rectangular parallelopiped in a given ellipsoid.

Let the equation to the ellipsoid be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, and let

x, y, and z be the half edges of the parallelopiped, then $u = 8xyz$.

But $z = c\left(1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}\right)$, $\therefore u = 8cxy\left(1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}\right)^{\frac{1}{2}}$;

$$\therefore \frac{du}{dx} = 8\,cy\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{1}{2}} - \frac{8\,cx^2y}{a^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-\frac{1}{2}} = 0;$$

$\therefore\ 1 - \dfrac{2\,x^2}{a^2} - \dfrac{y^2}{b^2} = 0.$ In like manner it appears that

$$1 - \frac{x^2}{a^2} - \frac{2\,y^2}{b^2} = 0.$$

$$2 - \frac{4\,x^2}{a^2} - \frac{2\,y^2}{b^2} = 0$$

$$1 - \frac{3\,x^2}{a^2} = 0 \quad \therefore\ 3\,x^2 = a^2 \text{ and } x = \frac{a}{\sqrt{3}},\ y = \frac{b}{\sqrt{3}},$$

$$z = \frac{c}{\sqrt{3}}.$$

Hence $u = \dfrac{8\,abc}{3\sqrt{3}}$.

(92.) Let $u = f(x, y, z)$, and let $u' = f(x + h, y + k, z + l)$, then when u is a maximum or minimum it can be demonstrated, as in (90), that $\dfrac{du}{dx} + m\,\dfrac{du}{dy} + n\,\dfrac{du}{dz} = 0$ if $k = mh$ and $l = nh$. But k and l are arbitrary, $\therefore$ m and n are also arbitrary; $\therefore$ this equation resolves itself into the three following: $\dfrac{du}{dx} = 0$, $\dfrac{du}{dy} = 0$, $\dfrac{du}{dz} = 0$.

Also, the equation of condition is

$$\left(\frac{d^2u}{dx^2}\,\frac{d^2u}{dy^2} - \left(\frac{d^2u}{dx\,dy}\right)^2\right)\left(\frac{d^2u}{dx^2}\,\frac{d^2u}{dz^2} - \left(\frac{d^2u}{dx\,dz}\right)^2\right) > \left(\frac{d^2u}{dy\,dz}\,\frac{d^2u}{dx^2} - \frac{d^2u}{dx\,dy}\,\frac{d^2u}{dx\,dz}\right)^2.$$

Vide Theorie des Fonctions, page 259.

Example $u = \dfrac{xyz}{(a+x)(x+y)(y+z)(z+b)}$.

$\log. u = \log. x + \log. y + \log. z - \log. (a+x) - \log. (x+y) - \log. (y+z) - \log. (z+b)$.

$$\frac{1}{u}\frac{du}{dx} = \frac{1}{x} - \frac{1}{a+x} - \frac{1}{x+y} = 0\,; \therefore ay - x^2 = 0, \text{ and } \frac{a}{x} = \frac{x}{y}.$$

$$\frac{1}{u}\frac{du}{dy} = \frac{1}{y} - \frac{1}{x+y} - \frac{1}{y+z} = 0\,; \therefore xz - y^2 = 0, \text{ and } \frac{x}{y} = \frac{y}{z}.$$

$$\frac{1}{u}\frac{du}{dz} = \frac{1}{z} - \frac{1}{y+z} - \frac{1}{z+b} = 0\,; \therefore by - z^2 = 0, \text{ and } \frac{y}{z} = \frac{z}{b}.$$

$$\therefore \frac{a^4}{x^4} = \frac{a}{x} \times \frac{x}{y} \times \frac{y}{z} \times \frac{z}{b} = \frac{a}{b}\,; \therefore x = \sqrt[4]{a^3 b},\ y = \sqrt[4]{a^2 b^2},$$

and $z = \sqrt[4]{ab^3}$. $\therefore u =$

$$\frac{\sqrt[4]{a^6 b^6}}{(a + \sqrt[4]{a^3 b})(\sqrt[4]{a^3 b} + \sqrt[4]{a^2 b^2})(\sqrt[4]{a^2 b^2} + \sqrt[4]{ab^3})(\sqrt[4]{ab^3} + b)}$$

$$= \frac{1}{a + 4\sqrt[4]{a^3 b} + 6\sqrt[4]{a^2 b^2} + 4\sqrt[4]{ab^3} + b} = \frac{1}{(a^{\frac{1}{4}} + b^{\frac{1}{4}})^4} \text{ a maximum.}$$

Examples for Practice.

(1.) Let $u = x^3 y^2 (a - x - y)$, then $u = \dfrac{a^6}{432}$ a maximum.

(2.) Let $u = xy + \dfrac{a^3}{x} + \dfrac{a^3}{y}$, then $u = 3a^2$ a minimum.

(3.) Let $u = xy\sqrt{a^2 b^2 - a^2 x^2 - b^2 y^2}$, then $u = \dfrac{a^2 b^2}{3\sqrt{3}}$ a maximum.

(4.) Let $u = a\,(\sin. x + \sin. y + \sin. (x + y))$, then $u = 3\,a\,\frac{\sqrt{3}}{2}$ a maximum.

(5.) Divide a number a into three such parts, that the continued product of the cube of the first, the fourth power of the second, and the fifth power of third, may be a maximum, $u = \frac{5^5\,a^{12}}{3^9\,4^8}$.

(6.) The perimeter of a triangle being given, determine its form when its area is a maximum.

Equilateral.

(7.) Of all rectangular parallelopipeds having a given volume, determine that which shall have the least surface.

A Cube.

(8.) Let $u = \cos. x \cos. y \cos. z$, and $x + y + z = \pi$, then $u = \frac{1}{8}$ a maximum.

(9.) Let $u = \sin. x \sin. y + \sin. x \sin. z + \sin. y \sin. z$ and $x + y + z = \frac{\pi}{4}$, then $u = \frac{3}{4}(2 - \sqrt{3})$ a maximum.

(10.) Let $u = ax\,y^2\,z^3 - x^2\,y^2\,z^3 - xy^3\,z^3 - xy^2\,z^4$, then $u = \left(\frac{a}{7}\right)^7$ a maximum.

CHAPTER XI.

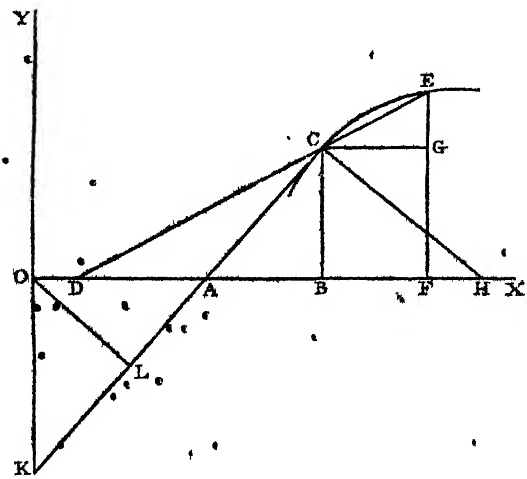
PLANE CURVES.

TANGENTS, NORMALS, AND ASYMPTOTES TO PLANE CURVES, REFERRED TO RECTILINEAR CO-ORDINATES.

Def. If a straight line cut a curve in two points, and the curve be made to revolve in its own plane about one of these points, until the other coincide with it, the straight line in its final position will be a tangent to the curve.

(93.) To find the angles which the tangent makes with the co-ordinate axes.

Let C E be a curve, O X and O Y rectangular axes of co-ordinates,



of which O is the origin, A C a tangent at C, D C E a secant. Draw C B and E F parallel to O Y, and C G to O X.

Let $OB = x$, $BC = y$, and $BF = h$, then, since $y = f(x)$, EF

$$= EG + y = y + \frac{dy}{dx} h + \frac{d^2y}{dx^2} h^2 + \&c. \therefore EG = \frac{dy}{dx} h + \frac{d^2y}{dx^2} \frac{h^2}{1.2} + \&c.$$

But $\tan. ECG = \frac{GE}{CG} = \dfrac{\frac{dy}{dx} h + \frac{d^2y}{dx^2} \frac{h^2}{1.2} + \&c.}{h} = \frac{dy}{dx} +$

$\frac{d^2y}{dx^2} \frac{h}{1.2} + \&c.$

Take the limits of both sides, and observe that when E coincides with C, DC will coincide with AC, and the angle ECG will be equal to CAB. $\therefore \tan. CAB = \frac{dy}{dx} = \cot. AKO$.

(94.) To find the subtangent AB and tangent AC.

(1.) $\frac{dy}{dx} = \frac{BC}{AB} = \frac{y}{AB}$; $\therefore AB = y \frac{dx}{dy} =$ subtangent.

(2.) $AC = \sqrt{BC^2 + AB^2} = \sqrt{y^2 + y^2 \frac{dx^2}{dy^2}} = y \sqrt{1 + \frac{dx^2}{dy^2}} =$ tangent.

(95.) To find the equation to the tangent AC.

Let the co-ordinates of the point C be x', y', and x, y any point in the line AC. The equation to a line which passes through a point x', y', and makes a given angle with the axis of x, is

$y - y' = m(x - x')$ (*Hymers's Conic Sections, page* 11). But $m = \frac{dy}{dx}$

$\therefore y - y' = \frac{dy}{dx}(x - x')$ is the equation to the tangent.

(96.) If CH be drawn through the point C at right angles to AC, it will be the normal, and BH the subnormal. To find these,

(1.) AB : BC :: BC : BH, $\therefore$ BH $= \dfrac{BC^2}{AB} = \dfrac{y^2}{y\dfrac{dx}{dy}} = y\dfrac{dy}{dx}$ = subnormal.

(2.) CH $= \sqrt{BC^2 + BH^2} = \sqrt{y^2 + y^2\dfrac{dy^2}{dx^2}} = y\sqrt{1 + \dfrac{dy^2}{dx^2}} =$ normal.

(97.) To find the equation to the normal CH.

Let the co-ordinates of C be x', y', then since CH is perpendicular to AC, its equation is $y - y' = -\dfrac{1}{m}(x - x')$.—(*Hymers's Conic Sections, page* 13.)

$$\therefore y - y' = -\frac{dx}{dy}(x - x').$$

(98.) To find the points where the tangent cuts the co-ordinate axes.

(1.) O A = O B − A B $= x - y\dfrac{dx}{dy} = x_{,}$.

(2.) O K = A O tan. O A K $= -y + x\dfrac{dy}{dx} = y_{,}$.

(99.) To find the perpendicular from the origin on the tangent.

Let the equation to A C be $y = mx + c$, then $p =$ the perpendicular from the origin is equal to $\dfrac{c}{\sqrt{1+m^2}}$. (*Hymers's Conic Sections*, p. 17.)

But the equation to A C is $y - y' = \dfrac{dy}{dx}(x - x')$; (95)

$$\therefore y = \frac{dy}{dx}x + y' - \frac{dy}{dx}x'; \quad \therefore c = y' - \frac{dy}{dx}x';$$

and $\therefore p = \dfrac{y' - \dfrac{dy}{dx}x'}{\sqrt{1 + \dfrac{dy^2}{dx^2}}} = \dfrac{y - mx}{\sqrt{1+m^2}}$, if the accents be effaced, and m restored.

(100.) The results of the seven preceding articles may be exhibited in a tabular form.

(1.) Tan. C A B or cot. A K O $= \dfrac{dy}{dx}$.

(2.) Tangent $= y\sqrt{1 + \dfrac{dx^2}{dy^2}}$.

(3.) Subtangent $= y\dfrac{dx}{dy}$.

(4.) Normal $= y\sqrt{1 + \dfrac{dy^2}{dx^2}}$.

(5.) Subnormal $= y\dfrac{dy}{dx}$.

(6.) The equation to the tangent is $y - y' = \dfrac{dy}{dx}(x - x')$.

(7.) The equation to the normal is $y - y' = -\dfrac{dx}{dy}(x - x')$.

(8.) The perpendicular from the origin on the tangent is $= \dfrac{y - mx}{\sqrt{1+m^2}}$ where $m = \dfrac{dy}{dx}$.

(9.) O A $= x_0 = x - y\dfrac{dx}{dy} = y\dfrac{dx}{dy} - x$, when x_0 is measured in an opposite direction from the origin.

(10.) O K $= y_0 = -y + x\dfrac{dy}{dx} = y - x\dfrac{dy}{dx}$, when y_0 is measured in an opposite direction.

EXAMPLE (1). To find the equation to the tangent in an ellipse.

$\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1$ is the equation to an ellipse, the centre being the origin. (*Hymers's Conic Sections*, page 61) $\therefore \frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$. But the equation to the tangent is $y' - y = \frac{dy}{dx}(x' - x)$; (100)

$$\therefore y' - y = -\frac{b^2 x x'}{a^2 y} + \frac{b^2 x^2}{a^2 y},$$

$$a^2 y y' - a^2 y^2 = -b^2 x x' + b^2 x^2,$$

$$a^2 y y' + b^2 x x' = a^2 y^2 + b^2 x^2 = a^2 b^2;$$

$$\therefore \frac{y y'}{b^2} + \frac{x x'}{a^2} = 1.$$

Ex. (2). In an ellipse to find the subtangent.

From Example (1) we have $y y' = \frac{b^2}{a^2}(a^2 - x x')$;

$\therefore y' = \frac{b}{a} \frac{a^2 - x x'}{\sqrt{a^2 - x^2}}$. Let $y' = 0$, then $a^2 - x x' = 0$;

$\therefore x' = \frac{a^2}{x}$. But the subtangent $= x' - x = \frac{a^2 - x^2}{x}$.

Ex. (3). The equation to a parabola referred to two tangents as axes is $\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1$: find the intercepts of the tangent along x and y.

Since $\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1$

$$\frac{dx}{2a\left(\frac{x}{a}\right)^{\frac{1}{2}}} + \frac{dy}{2b\left(\frac{y}{b}\right)^{\frac{1}{2}}} = 0; \quad \therefore \frac{dy}{dx} = -\left(\frac{by}{ax}\right)^{\frac{1}{2}}.$$

But $OA = x_0 = x - y\dfrac{dx}{dy} = x + y\left(\dfrac{ax}{by}\right)^{\frac{1}{2}} = x + a^{\frac{1}{2}}x^{\frac{1}{2}}\left(\dfrac{y}{b}\right)^{\frac{1}{2}} =$

$x + (ax)^{\frac{1}{2}}\left(1 - \left(\dfrac{x}{a}\right)^{\frac{1}{2}}\right) = (ax)^{\frac{1}{2}}$.

In a similar manner it appears that $OK = y_0 = (by)^{\frac{1}{2}}$.

Ex. (4). The equation to the cissoid of Diocles is $y = \dfrac{x^{\frac{3}{2}}}{(2a-x)^{\frac{1}{2}}}$: find the subtangent and subnormal.

Since $y = \dfrac{x^{\frac{3}{2}}}{(2a-x)^{\frac{1}{2}}}$, $\dfrac{dy}{dx} = \dfrac{x^{\frac{1}{2}}(3a-x)}{(2a-x)^{\frac{3}{2}}}$.

But the subtangent $= y\dfrac{dx}{dy} = \dfrac{y(2a-x)^{\frac{3}{2}}}{x^{\frac{1}{2}}(3a-x)} = \dfrac{x(2a-x)}{3a-x}$.

Again, the subnormal $= y\dfrac{dy}{dx} = \dfrac{x^{\frac{3}{2}}}{(2a-x)^{\frac{1}{2}}}\,\dfrac{x^{\frac{1}{2}}(3a-x)}{(2a-x)^{\frac{3}{2}}} =$
$\dfrac{x^2(3a-x)}{(2a-x)^2}$.

Ex. (5). The equation to the catenary is $y = \dfrac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)$: find the normal and subnormal.

Since $y = \dfrac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)$, $\dfrac{dy}{dx} = \dfrac{1}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right)$.

But the normal $= y\left(1 + \left(\dfrac{dy}{dx}\right)^2\right)^{\frac{1}{2}} = y\left(1 + \dfrac{1}{4}\left(e^{\frac{2x}{c}} - 2 + e^{-\frac{2x}{c}}\right)\right)^{\frac{1}{2}}$

$= \dfrac{y}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right) = \dfrac{y^2}{c}$.

Again, the subnormal $= y\dfrac{dy}{dx} = \dfrac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right) \times \dfrac{1}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right)$

$= \dfrac{c}{4}\left(e^{\frac{2x}{c}} - e^{-\frac{2x}{c}}\right)$.

(101.) Since an arc and its tangent coincide at the point of contact, the angles at which the curve cuts the co-ordinate axes may be found.

Let $\mathfrak{x}$ and $\mathfrak{y}$ represent the angles at which the curve cuts the axes of x and y respectively; then $\tan. \mathfrak{x} = \frac{dy}{dx}$, and $\tan. \mathfrak{y} = \frac{dx}{dy}$.

EXAMPLE. Let $y = \frac{b}{a}(a^2 - x^2)^{\frac{1}{2}}$, which is the equation to the ellipse when the centre is the origin, to find at what angles it cuts the axes of x and y.

$\frac{dy}{dx} = -\frac{bx}{a(a^2 - x^2)^{\frac{1}{2}}}$, from which if $x = \pm a$, $\tan. \mathfrak{x} = \mp \frac{b}{0} = \mp \infty = \tan. 90^\circ$; $\therefore \mathfrak{x} = 90^\circ$, which is the angle at which the curve cuts the axis of x at the distance of $\pm a$ from the origin.

Again, let $x = 0$, then $\tan. \mathfrak{y} = \frac{dx}{dy} = -\frac{a^2}{0} = \infty$; $\therefore \tan. \mathfrak{y} = \tan. 90^\circ$; $\therefore \mathfrak{y} = 90^\circ$. $\therefore$ the curve cuts the axis of y at the same angle.

(102.) To find the angle at which two curves, whose equations are $y = f(x)$, and $y' = \varphi(x')$, intersect each other.

Let the curves be referred to the same rectangular axes; and let θ be the angle made by two tangents at the point of intersection, and $\mathfrak{x}$ and $\mathfrak{x}'$ the angles which these tangents make with the axis of x. Then θ will be the angle required, and $\tan. \theta = \tan. (\mathfrak{x} - \mathfrak{x}')$

$$= \frac{\tan. \mathfrak{x} - \tan. \mathfrak{x}'}{1 + \tan. \mathfrak{x} \tan. \mathfrak{x}'} = \frac{\frac{dy}{dx} - \frac{dy'}{dx'}}{1 + \frac{dy}{dx} \cdot \frac{dy'}{dx'}}.$$

EXAMPLE (1.) Let a circle whose equation is $y^2 = 2ax - x^2$ be cut

by a straight line whose equation is $y' = x'$: find the angle at the point of intersection.

At the point of intersection $x = x'$, and $y = y'$; $\therefore 2\,ax - x^2 = x^2$, and $x = a = y$.

But (1) $\frac{dy}{dx} = \frac{a - x}{y}$, and (2) $\frac{dy}{dx} = 1$; $\therefore$ tan. $\theta = \frac{a - x - y}{y + a - x} =$

$- \frac{a}{a} = -1$. $\therefore \theta = 135°$.

Ex. (2.) The equation to a parabola is $y^2 = 4\,ax$, and to a circle

$y'^2 = a'^2 - x'^2$. If $a = \frac{a'}{2}$, find the point where the curves intersect.

At the point of intersection $x = x'$, and $y = y'$, and $2\,a'\,x = a'^2 - x^2$

$\therefore a' + x = a' \sqrt{2}$, and $x = a' (\sqrt{2} - 1)$, also $y = a' \sqrt{2} \sqrt{\sqrt{2} - 1}$.

But (1) $\frac{dy}{dx} = \frac{a'}{y}$; and (2) $\frac{dy}{dx} = - \frac{x}{y}$. $\therefore$ tan. $\theta = \frac{y\,(a' + x)}{y^2 - a'\,x}$

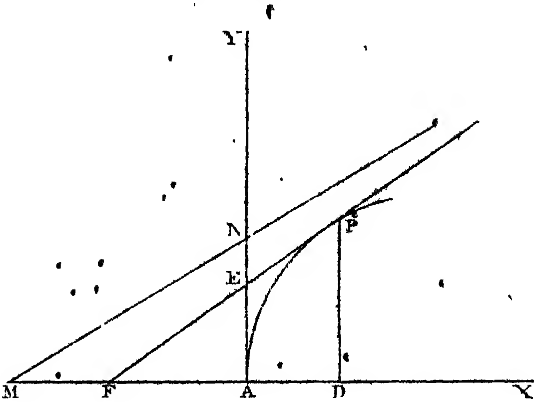
$= \frac{2}{\sqrt{\sqrt{2} - 1}}$.

ASYMPTOTES.

(103. Let A P be a curve, and let A be the origin of co-ordinates, and A X and A Y the axes of x and y.

Let F P be a tangent at the point P (x', y').

Then A F $= y' \frac{dx'}{dy'} - x'$,



and $A\,E = y' - \frac{dy'}{dx'}\,x'$. . . (100.)

Let x' or y', or both, become now infinite, and A F and A E remain finite, then F P will meet the curve only at an infinite distance. It is therefore a *rectilinear asymptote* to the curve.

Let A M and A N be the values of A F and A E when x' or y', or both, are increased indefinitely, then two points, M and N, in the asymptote are found, and therefore M N produced is the line required.

EXAMPLE (1.) $y^2 = mx + nx^2$ is the equation to the conic sections when the latus rectum is m.

Hence $\frac{dy}{dx} = \frac{m + 2\,nx}{2\,y}$. $\therefore$ A F $= \frac{2\,y^2}{m + 2\,nx} - x = \frac{mx}{m + 2\,nx}$;

and A E $= y - \frac{mx + 2\,nx^2}{2\,y} = \frac{mx}{2\sqrt{mx + nx^2}}$. $\therefore$ A F $= \frac{m}{\frac{m}{x} + 2\,n}$,

and A E $= \frac{m}{2\sqrt{\frac{m}{x} + n}}$. Let x become infinite, then A M $= \frac{m}{2\,n}$,

and A N $= \frac{m}{2\sqrt{n}}$.

(1.) If n be $= 0$, as is the case in the parabola, then $AN = \infty$; $\therefore$ the parabola has no rectilinear asymptote.

(2.) If n be negative, as is the case in the ellipse, AN is imaginary; and therefore the ellipse does not admit of a rectilinear asymptote.

(3.) If n be positive, which it is in the hyperbola, AN has a finite value; and therefore this curve admits of a rectilinear asymptote.

Ex. (2.) To draw an asymptote to a hyperbola:

$$y^2 = \frac{b^2}{a^2}(2ax + x^2) \therefore \frac{dy}{dx} = \frac{b^2(a + x)}{a^2 y}$$

$$\text{But } AE = y - \frac{dy}{dx}x = \frac{b^2 x}{ay} = \frac{bx}{\sqrt{2ax + x^2}} = \frac{b}{\sqrt{\frac{2a}{x} + 1}} = b,$$

when $x = \infty$

$$\text{and } AF = -x + y\frac{dx}{dy} = -x + \frac{2ax + x^2}{a + x} = \frac{ax}{a + x} = \frac{a}{\frac{a}{x} + 1}$$

$= a$, when $x = \infty$.

$\therefore AM = a$, and $AN = b$, join MN and produce it; it is the asymptote required.

The values of AM and AN might have been obtained from Ex. (1.) as follows:—

$$\text{Since } y^2 = \frac{b^2}{a^2}(2ax + x^2) = \frac{2b^2}{a}x + \frac{b^2}{a^2}x^2. \quad \therefore m = \frac{2b^2}{a}, \text{ and}$$

$$n = \frac{b^2}{a^2}. \therefore AM = \frac{m}{2n} = a, \text{ and } AN = \frac{m}{2\sqrt{n}} = b.$$

Ex. (3). To determine whether the cissoid of Diocles admits of an asymptote

$$y = \frac{x^{\frac{3}{2}}}{(2a - x)^{\frac{1}{2}}}; \therefore \frac{dy}{dx} = \frac{x^{\frac{1}{2}}(3a - x)}{(2a - x)^{\frac{3}{2}}}.$$

Hence $AF = y\frac{dx}{dy} - x = \frac{-ax}{3a-x}$ and $AE = y - \frac{x\,dy}{dx} = -a\left(\frac{x}{2a-x}\right)^{\frac{3}{2}}$.

Let $x = 2a$, then y and $\frac{dy}{dx}$ become infinite. $\therefore$ there is an asymptote to the curve perpendicular to the axis of x at the distance $2a$ from the origin.

(104.) To find the equation to the rectilineal asymptote to a curve.

It appears from (95) that $y - y' = \frac{dy}{dx}(x - x')$ is the equation to the tangent; and if in this x' or y', or both, be made infinite, the equation to the asymptote is determined, and hence the line itself may be drawn; but this method is in general very complex, and the following is therefore usually adopted:—If the equation to the curve can be reduced to the form $y = ax + b + \frac{c}{x} + \frac{d}{x^2} + \&c.$, then $y = ax + b$ is the equation to the asymptote

For as x increases, $\frac{c}{x} + \frac{d}{x^2} + \&c.$ diminish; and when x becomes indefinitely great, these terms become indefinitely small; and therefore the straight line whose equation is $y = ax + b$ meets the curve only at an infinite distance from the origin: it is therefore an asymptote to it

EXAMPLE (1.) Find the equation to the asymptote to a hyperbola.

$$y = \pm\frac{b}{a}(x^2 - a^2)^{\frac{1}{2}} = \pm\frac{bx}{a}\left(1 - \frac{1}{2}\frac{a^2}{x^2} - \frac{1}{8}\frac{a^4}{x^4} - \&c.\right) =$$

$$\pm\frac{bx}{a} \mp \frac{ab}{2x} \mp \frac{a^3b}{8x^3} \mp \&c. \quad \text{Let } x = \infty, \text{ then } y = \pm\frac{bx}{a}.$$

$\therefore$ the hyperbola has two asymptotes which pass through the origin and make angles with the axis of x, whose tangents are $+\frac{b}{a}$ and

Ex. (2). Find the equation to the asymptote of the curve whose equation is $y^3 = ax^2 - x^3$.

$y = -x\left(1 - \frac{a}{x}\right)^{\frac{1}{3}} = -x + \frac{a}{3} + \frac{1}{9}\frac{a^2}{x} + \&c.$ Let $x = \infty$, then

$y = -x + \frac{a}{3}$, which is the equation to the asymptote.

(105.) If the equation to a curve can be reduced to the form $y = ax^2 + bx + c + \frac{d}{x} + \frac{e}{x^2} + \&c.$, then it admits of a *parabolic* asymptote whose equation is $y = ax^2 + bx + c$.

Ex. Let the equation to a curve be $x^3 - ay(x - b) = 0$: to determine whether it admits of an asymptote; and if so, of what kind.

Since $x^3 - ay(x - b) = 0$, $ay = \frac{x^3}{x - b} = x^2 + bx + b^2 + \frac{b^3}{x} + \frac{b^4}{x^2} + \&c.$ Let $x = \infty$, then $y = \frac{1}{a}x^2 + \frac{b}{a}x + \frac{b^2}{a}$; $\therefore$ the curve admits of a parabolic asymptote.

Examples for Practice.

(1.) If $y = \frac{b^2 x}{a^2 + x^2}$ be the equation to a curve, prove that it cuts the axis of x at the origin at an angle $= \tan.^{-1}\frac{b^2}{a^2}$.

(2.) If $y^3 + ay = ax - x^2$ be the equation to a curve, prove that it cuts the axis of x at the distance of a from the origin at an angle of 45°.

(3.) The equation to a curve is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$: find the equation to its tangent. Ans. $\frac{x'}{x^{\frac{1}{3}}} + \frac{y'}{y^{\frac{1}{3}}} = a^{\frac{2}{3}}$.

(4.) The equation to the logarithmic curve is $y = a^x$: find the subtangent. Ans. $\dfrac{1}{\log. a}$.

(5.) The equation to a hyperbola is $a^2 y^2 - b^2 x^2 = - a^2 b^2$: find the equation to its tangent. Ans. $a^2 y\, y' - b^2 x\, x' = - a^2 b^2$.

(6.) The equation to a tangent to the equilateral hyperbola is $y = mx + a \sqrt{m^2 - 1}$, and the equation to the perpendicular on it from the centre is $y = - \dfrac{1}{m} x$: find the locus of their intersection. Ans. $(y^2 + x^2)^2 = a^2 (x^2 - y^2)$, which is the lemniscata of Bernoulli.

(7.) The equation to a curve is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$: prove that the length of the tangent intercepted between the co-ordinate axes is invariable.

(8.) The locus of the intersection of a tangent to a parabola with a perpendicular on it from the vertex is the cissoid of Diocles.

(9.) If two tangents to a parabola intersect at right angles, the locus of their intersection is the directrix.

(10.) If two tangents to an ellipse intersect at right angles, the locus of their intersection is a circle whose radius $= \sqrt{a^2 + b^2}$.

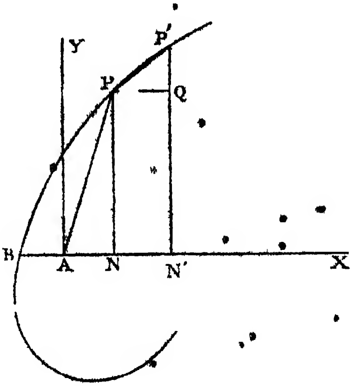
(11.) If the equation to a curve be $x^3 + y^3 = a^3$, prove that the equation to a rectilineal asymptote is $x + y = 0$.

(12.) If $y^3 - 2xy^2 + x^2 y - a^3 = 0$ be the equation to a curve, prove that the equations to its asymptotes are $y = x$ and $y = 0$.

CHAPTER XII.

ARCS, AREAS, TANGENTS, NORMALS, AND ASYMPTOTES TO CURVES REFERRED TO POLAR CO-ORDINATES.

(106.) To find the differential of the arc of a plane curve as a function of the rectangular co-ordinates of its extremity.



It appears from the equation to the curve that $y = f(x)$.

Let $A N = x$, $N P = y$, and $N N' = h$.

Then we have by Taylor's theorem

$$P'Q = \frac{dy}{dx} h + \frac{d^2y}{dx^2} \frac{h^2}{1 \cdot 2} + \ldots,$$

But $P P' = \sqrt{P Q^2 + P' Q^2}$

$$= \sqrt{h^2 + \frac{dy^2}{dx^2} h^2 + A h^3 + B h^4 + \ldots}$$

$$\therefore \frac{P P'}{h} = \sqrt{1 + \frac{dy^2}{dx^2} + A h + B h^2 + \ldots}$$

Taking the limits of both sides, we have $\frac{ds}{dx} = \sqrt{1 + \frac{dy^2}{dx^2}}$,

$\therefore ds = \sqrt{dx^2 + dy^2}$

(107.) To find the differential of an arc in terms of the polar co-ordinates of its extremities.

Let A, the origin of co-ordinates, be the pole; and let A P $= r$, and the angle P A N $= \theta$; then $x = r$ cos. θ, and $y = r$ sin. θ,

$$\therefore \frac{dx}{d\theta} = \frac{dr}{d\theta} \cos.\ \theta - r \sin.\ \theta, \text{ and } \frac{dy}{d\theta} = \frac{dr}{d\theta} \sin.\ \theta + r \cos.\ \theta;$$

$\therefore dx^2 + dy^2 = r^2\, d\theta^2 + dr^2$. But $ds = \sqrt{dx^2 + dy^2}$, (106)

$\therefore ds = \sqrt{r^2\, d\theta^2 + dr^2}$.

(108.) To find the differential of the area of a plane curve as a function of its rectangular co-ordinates.

Let B N P $= a$, and let N N' $= h$, then P N N' P' $= \dfrac{y + y + \frac{dy}{dx} h + \ldots}{2} \times h$

$\therefore \dfrac{\text{P N N' P'}}{h} = y + \dfrac{dy}{dx} \dfrac{h}{1.2} + \ldots$ Taking the limits of both sides, we have $\dfrac{da}{dx} = y$, $\therefore da = y\, dx$.

(109.) To find the differential of a plane curve surface as a function of its polar co-ordinates.

Let B A P $= a$, the angle B A P $= \theta$, and A P $= r$, then

$$a = \text{B N P} - \text{A N P} = \text{B N P} - \tfrac{1}{2} xy; \therefore \frac{da}{d\theta} = d\,.\,\frac{\text{B N P}}{d\theta} - \tfrac{1}{2} d\, \frac{xy}{d\theta}$$

$= y \dfrac{dx}{d\theta} - \tfrac{1}{2}\left(y \dfrac{dx}{d\theta} + x \dfrac{dy}{d\theta}\right) = \tfrac{1}{2}\left(y \dfrac{dx}{d\theta} - x \dfrac{dy}{d\theta}\right)$. But $x = - r$ cos. θ,

and $y = r$ sin. θ, $\therefore y \dfrac{dx}{d\theta} = - r \sin.\ \theta \cos.\ \theta \dfrac{dr}{d\theta} + r^2 \sin.^2 \theta$;

$$\text{and } x\frac{dy}{d\theta} = -r\sin.\theta\cos.\theta\frac{dr}{d\theta} - r^2\cos.^2\theta:$$

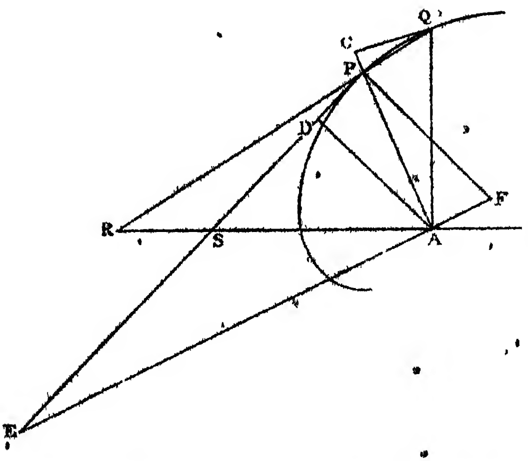
$$\therefore \frac{da}{d\theta} = \frac{1}{2}\left(y\frac{dx}{d\theta} - x\frac{dy}{d\theta}\right) = \frac{1}{2}r^2, \text{ and } da = \frac{1}{2}r^2 d\theta.$$

(110.) To find the angle which a tangent to a curve makes with the radius vector.

Let A be the pole, A P the radius vector $= r$, A Q $= r'$, the angle R A P $= \theta$, and P A Q $= h$.

Then since r' is a function of r and θ, we have by Taylor's Theorem

$$r' - r = \frac{dr}{d\theta}\frac{h}{1} + \frac{d^2r}{d\theta^2}\frac{h^2}{1\,.\,2} + \dots\dots$$



Draw QC perpendicular to AP, then the angle A P R = QPC; ∴

$$\tan.\ \text{A P R} = \frac{\text{Q C}}{\text{P C}} = \frac{\text{Q C}}{\text{A C} - \text{A P}} = \frac{r'\sin.h}{r'\cos.h - r} = \frac{r'}{\dfrac{r'-r}{\sin.h} - r'\tan.\frac{1}{2}h}$$

$$= \frac{r + \dfrac{dr}{d\theta}\dfrac{h}{1} + \dots}{\dfrac{dr}{d\theta}\dfrac{h}{\sin.h} + \dots - (r + \dfrac{dr}{d\theta}\dfrac{h}{1} + \dots)\tan.\frac{1}{2}h},$$ Let Q now move

down to P, then RP shall coincide with SP, and become a tangent at P, and the angle APR shall become equal to APS. Therefore

$$(1.)\ \tan. APS = \frac{r}{\frac{dr}{d\theta}} = \frac{r\,d\theta}{dr} = \tan. P.$$

$$(2.)\ \sin.^2 P = \frac{\tan.^2 P}{1 + \tan.^2 P} = \frac{\frac{r^2 d\theta^2}{dr^2}}{1 + \frac{r^2 d\theta^2}{dr^2}} = \frac{r^2 d\theta^2}{dr^2 + r^2 d\theta^2};$$

$$\therefore\ \sin. P = \frac{r\,d\theta}{\sqrt{dr^2 + r^2 d\theta^2}}.$$

$$(3.)\ \cos.^2 P = \frac{1}{1 + \tan.^2 P} = \frac{1}{1 + \frac{r^2 d\theta^2}{dr^2}} = \frac{dr^2}{dr^2 + r^2 d\theta^2};$$

$$\therefore\ \cos. P = \frac{dr}{\sqrt{dr^2 + r^2 d\theta^2}}.$$

(111.) To find the perpendicular from the pole on the tangent and the intercept on the tangent.

Draw AD perpendicular to PS, then

$$(1.)\ AD = r \sin. P = \frac{r^2 d\theta}{\sqrt{dr^2 + r^2 d\theta^2}} = \frac{r^2}{\sqrt{r^2 + \frac{dr^2}{d\theta^2}}} = p.$$

$$(2.)\ PD = r \cos. P = \frac{r\,dr}{\sqrt{dr^2 + r^2 d\theta^2}} = \frac{r}{\sqrt{1 + r^2\frac{d\theta^2}{dr^2}}} =$$

$$\frac{r\frac{dr}{d\theta}}{\sqrt{r^2 + \frac{dr^2}{d\theta^2}}} = \frac{r\,dr}{ds}. \quad \ldots \ (107.)$$

$$\text{Cor. } \frac{d\theta}{dr} = \frac{p}{r\sqrt{r^2 - p^2}}; \text{ and } \frac{ds}{dr} = \frac{r}{\sqrt{r^2 - p^2}}.$$

Now, when r is a maximum or minimum, $\frac{dr}{d\theta} = 0$; $\therefore$ the greatest and least distances of the curve from the pole are easily found.

(112.) To find the tangent and subtangent.

Through A draw AE perpendicular to AP, meeting PS in E, then PE is the magnitude of the tangent, and AE of the subtangent.

$$(1.)\ \mathrm{PE} = \frac{\mathrm{AP}}{\cos.\ \mathrm{P}} = r\sqrt{1 + r^2\frac{d\theta^2}{dr^2}}.$$

$$(2.)\ \mathrm{AE} = \mathrm{AP}.\ \tan.\ \mathrm{P} = \frac{r^2 d\theta}{dr}.$$

(113.) To find the normal and subnormal.

Draw PF perpendicular to EP, meeting EA in F, then PF is the normal, and AF the subnormal.

$$(1.)\ \mathrm{PF} = \frac{\mathrm{PA}}{\sin.\ \mathrm{P}} = \sqrt{r^2 + \frac{dr^2}{d\theta^2}}$$

$$(2.)\ \mathrm{AF} = \frac{\mathrm{AP}}{\tan.\ \mathrm{P}} = \frac{dr}{d\theta}.$$

EXAMPLE (1.) To find the angle which the tangent makes with the radius vector in the spiral of Archimedes, whose equation is $r = a\theta$.

$$\therefore \frac{1}{a} = \frac{d\theta}{dr}, \text{ and } \tan.\ \mathrm{P} = \frac{r\,d\theta}{dr} = \frac{r}{a}.$$

Ex. (2.) To find the magnitude of the tangent and subtangent in the logarithmic spiral, whose equation is $r = a^\theta$.

$$\log.\ r = \theta \log.\ a;$$

$$\therefore \frac{dr}{r} = \log\ a\,d\theta \quad \therefore \frac{1}{\log.\ a} = r\frac{d\theta}{dr};$$

$$\therefore PE = r\sqrt{1+\frac{r^2 d\theta^2}{dr^2}} = \frac{r}{\log. a}\sqrt{1+\log.^2 a}.$$

$$\text{And } AE = \frac{r^2 d\theta}{dr} = \frac{r}{\log. a}.$$

Ex. (3.) To find the perpendicular from the pole on the tangent in the curve, whose equation is $r = a \sin. 2\theta$.

$$\frac{dr}{d\theta} = 2a\cos. 2\theta = 2a\sqrt{1-\sin.^2 2\theta} = 2a\frac{\sqrt{a^2-r^2}}{a} = 2\sqrt{a^2-r^2}$$

$$\therefore AD = p = \frac{r^2}{\sqrt{r^2+\frac{dr^2}{d\theta^2}}} = \frac{r^2}{\sqrt{4a^2-3r^2}}.$$

Ex. (4.) To find the magnitude of the normal in the curve whose equation is $a = r\cos. \theta$.

$$r = \frac{a}{\cos. \theta} \quad \therefore \frac{dr}{d\theta} = \frac{a\sin. \theta}{\cos.^2 \theta} = \frac{r}{a}\sqrt{r^2-a^2};$$

$$\therefore PF = \sqrt{r^2+\frac{dr^2}{d\theta^2}} = \sqrt{r^2+\frac{r^2}{a^2}(r^2-a^2)} = \frac{r^2}{a}.$$

Ex. (5.) To find the magnitude of the subnormal in the curve whose equation is $re^{\theta} = a + \sqrt{a^2-r^2}$

$$\frac{d\theta}{dr} = -\frac{a}{r\sqrt{a^2-r^2}}; \quad \therefore AF = \frac{dr}{d\theta} = -\frac{r\sqrt{a^2-r^2}}{a}$$

(114.) To determine when a curve referred to polar co-ordinates admits of an asymptote.

From the equation to the curve we have $\theta = f(r)$ and the subtan-

gent AE $= r^2 \frac{d\theta}{dr}$. If, therefore, a particular value of θ render r infinite, and at the same time $r^2 \frac{d\theta}{dr}$ finite or equal to zero, a straight line drawn through the extremity of the subtangent parallel to the radius vector r, is an asymptote to the curve.

If a finite value of r render θ infinite, the curve admits of an asymptotic circle.

EXAMPLE (1.) To determine whether the reciprocal spiral whose equation is $r = a\,\theta^{-1}$, admits of an asymptote.

Since $r = a\,\theta^{-1}$, $\theta = \frac{a}{r}$, and when $r = \infty$, $\theta = 0$.

Also the subtangent $= r^2 \frac{d\theta}{dr} = -a$. If, therefore, A E be drawn $= -a$, and from its extremity a parallel be drawn to the radius vector, it will be an asymptote to the curve.

Ex. (2.) To determine whether the curve whose equation is $r^2 \sin. 2\theta = 2a^2$, admits of an asymptote.

$\sin. 2\theta = \frac{2a^2}{r^2} = 0$ when $r = \infty$. Also the subtangent $= r^2 \frac{d\theta}{dr}$

$$= -\frac{2a^2}{r \cos. 2\theta} = -\frac{2a^2}{\sqrt{r^2 - \frac{4a^4}{r^2}}} = 0 \text{ when } r = \infty;\ \therefore \text{ the radius}$$

vector, when $2\theta = 0$, or $2\theta = \pi$, is an asymptote to the curve.

Examples for Practice.

(1.) In the curve whose equation is $r^2 = a^2 \tan. 2\theta$, find the angle which the tangent makes with the radius vector.

$$\tan. P = \frac{a^2 r^2}{a^4 + r^4}.$$

(2.) In the Lemniscata of Bernoulli, whose equation is $r^2 = a^2 \cos. 2\theta$, find the perpendicular from the pole on the tangent.

$$p = \frac{r^3}{a^2}.$$

(3.) In the curve whose equation is $r = a\,(e^{c+\theta} + e^{c-\theta})$, find the perpendicular from the pole on the tangent.

$$p = \frac{r^2}{\sqrt{2r^2 - 4a^2 e^{2c}}}$$

(4.) In the spiral of Archimedes, whose equation is $r = a\,\theta$, find the magnitude of the subtangent.

$$\text{The subtangent} = \frac{r^2}{a}.$$

(5.) The equation to a circle referred to a point in its circumference is $r = a \cos\ \theta$, find the magnitude of the tangent.

$$\text{The tangent} = \frac{ar}{\sqrt{a^2 - r^2}}.$$

(6.) In the logarithmic spiral whose equation is $r = c\,e^{\frac{\theta}{a}}$, find the subtangent.

$$\text{The subtangent} = r\,a.$$

(7.) In the curve whose equation is $r = \dfrac{2a}{e^{\theta} + e^{-\theta}}$, find the magnitude of the normal.

$$\text{The normal} = \frac{r}{a}\sqrt{2a^2 - r^2}.$$

(8.) In the curve whose equation is $r = a - b\,\theta^{\frac{1}{2}}$, find the magnitude of the subnormal.

The subnormal $= -\frac{b^2}{2\,(a - r)}$.

(9.) To find whether the curve whose equation is $r \cos. \theta = a \cos. 2\,\theta$ admits of an asymptote.

The asymptote is perpendicular to the line from which θ is measured, and the subtangent $= -\,a$.

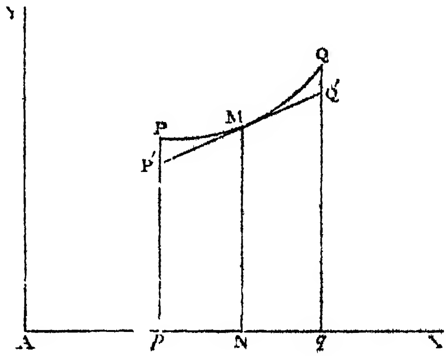
(10.) To find whether the curve whose equation is $(r - a)\,\theta = \sqrt{a^2\,\theta^2 - 1}$ admits of an asymptote.

It admits of a circular asymptote having $r = 2\,a$.

CHAPTER XIII.

THE DIRECTION OF CURVATURE, OSCULATING CURVES, THE RADIUS OF CURVATURE, INVOLUTES, AND EVOLUTES.

(115.) To find the direction of curvature of a curve referred to rectangular co-ordinates.



Let P M Q be a curve, and let A N = x, M N = y, p N = q N = h, and P' Q' a tangent at the point M; then

$$Q'q = y + \frac{dy}{dx} h$$

$$P'p = y - \frac{dy}{dx} h$$

$$Qq = y + \frac{dy}{dx} h + \frac{d^2y}{dx^2} \frac{h^2}{1 \cdot 2} + \frac{d^3y}{dx^3} \frac{h^3}{1 \cdot 2 \cdot 3} + \&c.$$

$$Pp = y - \frac{dy}{dx} h + \frac{d^2y}{dx^2} \frac{h^2}{1 \cdot 2} - \frac{d^3y}{dx^3} \frac{h^3}{1 \cdot 2 \cdot 3} + \&c.$$

$$QQ' = \frac{d^2y}{dx^2} \frac{h^2}{1 \cdot 2} + \frac{d^3y}{dx^3} \frac{h^3}{1 \cdot 2 \cdot 3} + \&c.$$

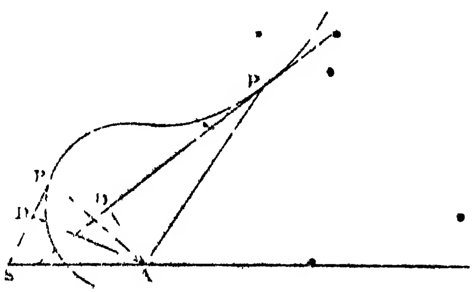
$$\text{and } PP' = \frac{d^2y}{dx^2} \frac{h^2}{1 \cdot 2} - \frac{d^3y}{dx^3} \frac{h^3}{1 \cdot 2 \cdot 3} + \&c.$$

But if h be taken very small, $\frac{d^2y}{dx^2} \frac{h^2}{1 \cdot 2}$ is greater than all the terms

which follow it in each of the above series (63), ∴ Q Q′ and P P′ have the same sign; and if $\frac{d^2y}{dx^2}$ be positive, the curve is convex to the axis of x; but if $\frac{d^2y}{dx^2}$ be negative, the curve is concave to the same axis.

(116.) To find the direction of curvature of a curve referred to polar co-ordinates.

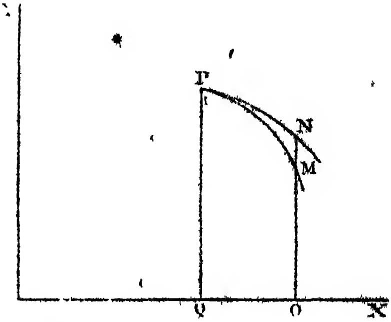
It is evident that if the curve be concave towards the pole, the perpendicular AD $= p$ increases or diminishes, as A P $= r$ the radius



vector increases or diminishes; but if the curve be convex to the pole, the perpendicular diminishes or increases as the radius vector increases or diminishes; therefore, when $p = f(r)$ is the equation to a curve, and it has its concavity towards the pole, $\frac{dp}{dr}$ is positive; but if its convexity be towards the pole, $\frac{dp}{dr}$ is negative It hence follows conversely, that if $\frac{dp}{dr}$ be positive, the curve is concave to the pole, and if $\frac{dp}{dr}$ be negative, it is convex.

(117.) To find the conditions necessary to the different orders of contact in osculating curves

Let P M and P N be two curves meeting at P, and referred to the same rectangular axes of co-ordinates, A X and A Y.



Let (x, y) be the co-ordinates of any point in P M, and $(x, y_{,})$ of any point in P N, then $y = f(x)$, and $y_{,} = \phi(x)$. At the point P, $y = y_{,}$.

Let Q O $= h$, then

$$\text{M O} = y + \frac{dy}{dx} h + \frac{d^2 y}{dx^2} \frac{h^2}{1 \cdot 2} + \frac{d^3 y}{dx^3} \frac{h^3}{1 \cdot 2 \cdot 3} + \&c.$$

$$\text{N O} = y + \frac{dy_{,}}{dx} h + \frac{d^2 y_{,}}{dx^2} \frac{h^2}{1 \cdot 2} + \frac{d^3 y_{,}}{dx^3} \frac{h^3}{1 \cdot 2 \cdot 3} + \&c.$$

$$\therefore \text{M N} = \left(\frac{dy_{,}}{dx} - \frac{dy}{dx}\right) h + \left(\frac{d^2 y_{,}}{dx^2} - \frac{d^2 y}{dx^2}\right) h^2 + \left(\frac{d^3 y_{,}}{dx^3} - \frac{d^3 y}{dx^3}\right)$$

$$\frac{h^3}{1 \cdot 2 \cdot 3} + \ldots = A_1 h + A_2 h^2 + A_3 h^3 + \ldots + A_n h^n + \ldots$$

If $A_1 = 0$, then $\frac{dy_{,}}{dx} = \frac{dy}{dx}$; $\therefore$ the curves have contact of the *first order.*

If $A_1 = 0$, and $A_2 = 0$, then $\frac{dy_{,}}{dx} = \frac{dy}{dx}$, and $\frac{d^2 y_{,}}{dx^2} = \frac{d^2 y}{dx^2}$; $\therefore$ the curves have contact of the *second order.*

If $A_1 = 0$, $A_2 = 0$, ... $A_n = 0$, then $\frac{dy_{,}}{dx} = \frac{dy}{dx}$, $\frac{d^2 y_{,}}{dx^2} = \frac{d^2 y}{dx^2}$

.... $\frac{d^n y_{,}}{dx^n} = \frac{d^n y}{dx^n}$: $\therefore$ the curves have contact of the n^{th} *order*

(118.) To determine from the equation to a curve, the order of contact which it may have with another given curve.

Let $y_{,} = \varphi(x_{,})$ be the equation to the curve, then, if it contain two constants, such values may be assigned to them that when $x_{,} = x$, $y_{,} = y$, and $\frac{dy_{,}}{dx} = \frac{dy}{dx}$, or that the curves may have contact of the first order. If $y_{,} = \varphi(x_{,})$ contain three constants, such values may be assigned to them that when $x_{,} = x$, $y_{,} = y$, $\frac{dy_{,}}{dx} = \frac{dy}{dx}$, and $\frac{d^2y_{,}}{dx^2} = \frac{d^2y}{dx^2}$, or that the curves may have contact of the second order; and if $y_{,} = \varphi(x_{,})$ contain $n + 1$ constants, such values may be assigned to them that the curves may have contact of the n^{th} order.

(1.) Let $y_{,} = ax_{,} + b$ be the equation to a straight line, and $y = f(x)$ the equation to a curve, then as the first equation contains two constants a and b, we may have contact of the first order. For this purpose, when $x_{,} = x$, $y_{,} = y$, $\frac{dy_{,}}{dx} = \frac{dy}{dx} = a$; $\therefore y = ax + b$, and $y_{,} - y = a(x_{,} - x) = \frac{dy}{dx}(x_{,} - x)$, which is the equation to the tangent at the point (x, y); $\therefore$ the tangent to a curve has contact of the first order.

(2.) Let $\rho^2 = (x_{,} - \alpha)^2 + (y_{,} - \beta)^2$ be the equation to a circle where ρ, α, and β are constants; then, in order that there may be contact of the second order, we must have $y_{,} = y$, $\frac{dy_{,}}{dx} = \frac{dy}{dx}$, and $\frac{d^2y_{,}}{dx^2} = \frac{d^2y}{dx^2}$, when $x_{,} = x$, from which three equations the constants ρ, α, and β may be determined. This circle is called the *circle of curvature*, and its radius the radius of curvature of any point (x, y) of a curve.

(119.) To find the radius of curvature at any point in a given curve.

Let $y = f(x)$ be the equation to the curve, and $\rho^2 = (x_{,} - \alpha)^2 + (y_{,} - \beta)^2$ the equation to the circle of curvature.

$$y_{,} = y, \frac{dy_{,}}{dx} = \frac{dy}{dx}, \text{ and } \frac{d^2y_{,}}{dx^2} = \frac{d^2y}{dx^2}, \text{ when } x_{,} = x.$$

$$\therefore (x - \alpha) + (y - \beta)\frac{dy}{dx} = 0.$$

$$1 + (y - \beta)\frac{d^2y}{dx^2} + \frac{dy^2}{dx^2} = 0.$$

Let $\frac{dy}{dx} = p$, and $\frac{d^2y}{dx^2} = q$;

Then $(x - \alpha)q + (y - \beta)pq = 0$,

$$p + (y - \beta)pq + p^3 = 0;$$

$$\therefore x - \alpha = \frac{p(1 + p^2)}{q},$$

$$\text{and } \alpha = x - \frac{p(1 + p^2)}{q}.$$

Also $y - \beta = -\frac{1 + p^2}{q}$,

$$\text{and } \beta = y + \frac{1 + p^2}{q}.$$

But $\rho^2 = (x - \alpha)^2 + (y - \beta)^2 = \frac{p^2(1 + p^2)^2}{q^2} + \frac{(1 + p^2)^2}{q^2} =$

$\frac{(1 + p^2)^3}{q^2}$; $\therefore \rho = \pm\frac{(1 + p^2)^{\frac{3}{2}}}{q} = -\frac{(1 + p^2)^{\frac{3}{2}}}{q}$.

*For if ρ be considered positive when the curve is concave to the axis of x, q is negative; and if ρ be considered negative when the curve is convex to the axis of x, q is positive.

Since, in the preceding investigation, both α and β, which are the co-ordinates of the centre of the circle of curvature are found, the circle itself is determined.

COR. (1). Since $\varrho = \dfrac{\left(1+\dfrac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{-\dfrac{d^2y}{dx^2}}$ is the expression for the radius of curvature when x is the independent variable, $\varrho = \dfrac{\left(1+\dfrac{dx^2}{dy^2}\right)^{\frac{3}{2}}}{\dfrac{d^2x}{dy^2}}$ is the corresponding expression when y is the independent variable.—Ex. (2) of (87).

COR. (2). Since $\dfrac{1}{\varrho} = \dfrac{-\dfrac{d^2y}{dx^2}}{\left(1+\dfrac{dy^2}{dx^2}\right)^{\frac{3}{2}}}$ when x is the independent variable; $\therefore \dfrac{1}{\varrho} = \dfrac{d^2x}{ds^2}\dfrac{dy}{ds} - \dfrac{d^2y}{ds^2}\dfrac{dx}{ds}$ when s is the independent variable.—Ex. (1) of (88). But $\dfrac{dx^2}{ds^2} + \dfrac{dy^2}{ds^2} = 1$. (106.) $\therefore \dfrac{d^2x}{ds^2}\dfrac{dx}{ds} + \dfrac{d^2y}{ds^2}\dfrac{dy}{ds} = 0$. $\therefore \dfrac{1}{\varrho^2} = \left(\dfrac{d^2x}{ds^2}\dfrac{dy}{ds} - \dfrac{d^2y}{ds^2}\dfrac{dx}{ds}\right)^2 + \left(\dfrac{d^2x}{ds^2}\dfrac{dx}{ds} + \dfrac{d^2y}{ds^2}\dfrac{dy}{ds}\right)^2$ $= \left(\left(\dfrac{d^2x}{ds^2}\right)^2 + \left(\dfrac{d^2y}{ds^2}\right)^2\right)\left(\dfrac{dx^2}{ds^2} + \dfrac{dy^2}{ds^2}\right) = \left(\dfrac{d^2x}{ds^2}\right)^2 + \left(\dfrac{d^2y}{ds^2}\right)^2$; $\therefore \dfrac{1}{\varrho} = \left(\left(\dfrac{d^2x}{ds^2}\right)^2 + \left(\dfrac{d^2y}{ds^2}\right)^2\right)^{\frac{1}{2}}$, and $\varrho = \dfrac{ds^2}{\sqrt{(d^2x)^2 + (d^2y)^2}}$, which is an expression for the radius of curvature when the arc is the independent variable.

COR. (3). Let r and θ be the polar co-ordinates of any point in a curve, and let θ be the independent variable. Let the origin of rectangular co-ordinates be taken as the pole, then $x = -r\cos.\theta$ and $y = r\sin\theta$, and since

$\rho = \dfrac{\left(1 + \dfrac{dx^2}{dy^2}\right)^{\frac{3}{2}}}{-\dfrac{d^2y}{dx^2}}$, it appears, as in Ex. (3) of (88), that

$$\rho = \frac{\left(r^2 + \dfrac{dr^2}{d\theta^2}\right)^{\frac{3}{2}}}{r^2 + 2\dfrac{dr^2}{d\theta^2} - r\dfrac{d^2r}{d\theta^2}}.$$

COR. (4.) $p = \dfrac{r^2}{\left(r^2 + \dfrac{dr^2}{d\theta^2}\right)^{\frac{1}{2}}}$ (111.) $\therefore \dfrac{dp}{dr} = \dfrac{r^3 - r^2\dfrac{d^2r}{d\theta^2} + 2r\dfrac{dr^2}{d\theta^2}}{\left(r^2 + \dfrac{dr^2}{d\theta^2}\right)^{\frac{3}{2}}}$;

$\therefore \dfrac{r\,dr}{dp} = \dfrac{\left(r^2 + \dfrac{dr^2}{d\theta^2}\right)^{\frac{3}{2}}}{r^2 + 2\dfrac{dr^2}{d\theta^2} - r\dfrac{d^2r}{d\theta^2}}$. Hence $\rho = \dfrac{r\,dr}{dp}$.

(120.) To find the locus of the centres of the circles of curvature of a curve referred to rectangular co-ordinates.

From (119) we have $\alpha = x - \dfrac{p(1+p^2)}{q}$, and $\beta = y + \dfrac{1+p^2}{q}$.

But α and β are the co-ordinates of the centre of the circle of curvature at the point (x, y). Therefore if by means of the above equations, and that to the curve x and y be eliminated, there will result an equation between α and β, which will be the equation of the locus required.

The curve whose co-ordinates are x and y, is called the *involute*, and that whose co-ordinates are α and β, the *evolute*. The reason of this will appear afterwards.

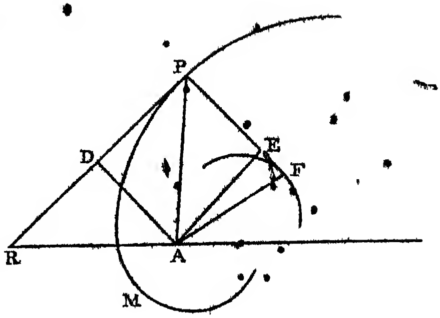
(121.) To find the evolute of a curve referred to polar co-ordinates.

Let P F be the radius of the circle of curvature at P, let A P $= r$, A F $= r'$, A D $= p$, and A E $= p'$, then

$$p' = \text{P D} = \sqrt{r^2 - p^2} \text{ and } r' = \sqrt{\text{A E}^2 + \text{E F}^2} = \sqrt{r^2 - p^2 + \left(\frac{r\,dr}{dp} - p\right)^2}$$

$$= \sqrt{r^2 - 2\,\frac{pr\,dr}{dp} + r^2\,\frac{dr^2}{dp^2}}.$$

Therefore, if from these two equations, and the equation to the curve PM, p and r be eliminated, there will result an equation between p' and r', which will be the equation to the evolute.



(122.) The radius of curvature at any point in a curve is a tangent to the evolute.

$$(y - \beta)^2 + (x - \alpha)^2 = \rho^2 \qquad (1)$$

$$(y - \beta)\frac{dy}{dx} + x - \alpha = 0 \qquad (2)$$

$$\text{and } (y - \beta)\frac{d^2y}{dx^2} + \frac{dy^2}{dx^2} + 1 = 0 \qquad (3)$$

By differentiating (2) upon the hypothesis that α and β are variable, we have

$$(y - \beta)\frac{d^2y}{dx^2} + \frac{dy^2}{dx^2} - \frac{dy}{dx}\cdot\frac{d\beta}{dx} + 1 - \frac{d\alpha}{dx} = 0. \quad (4)$$

Subtracting (4) from (3), we have

$$\frac{d\beta}{dx}\,\frac{dy}{dx} + \frac{d\alpha}{dx} = 0;$$

$$\therefore \frac{dy}{dx} = -\frac{d\alpha}{d\beta};$$

$$\therefore \beta - y = \frac{d\beta}{d\alpha}(\alpha - x),$$

$$\text{and } \beta - y = -\frac{1}{\frac{dy}{dx}}(\alpha - x).$$

Therefore x and y are co-ordinates of a point in the tangent to the evolute, through the point (α, β) and α and β are the co-ordinates of a point in the normal to the involute through the point (x, y); $\therefore$ the normal or radius of curvature at any point of the involute is a tangent to the evolute.

(123.) The radius of curvature of a curve, and the arc of its evolute, increase or decrease by equal differences.

$$\rho = \frac{(1+p^2)^{\frac{3}{2}}}{q} \ (119); \quad \therefore \frac{d\rho^2}{dx^2} = 9(1+p^2)p^2.$$

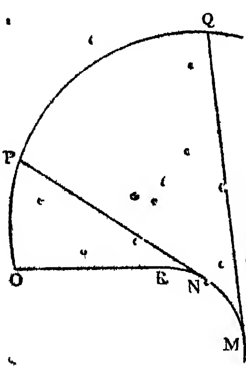
$$\alpha = x - \frac{p(1+p^2)}{q}; \quad \therefore \frac{d\alpha^2}{dx^2} = 9p^4.$$

$$\beta = y + \frac{1+p^2}{q}; \quad \therefore \frac{d\beta^2}{dx^2} = 9p^2.$$

$$\therefore \frac{d\alpha^2}{dx^2} + \frac{d\beta^2}{dx^2} = 9(1+p^2)p^2 = \frac{d\rho^2}{dx^2};$$

$$\therefore \frac{d\rho}{dx} = \sqrt{\frac{d\alpha^2}{dx^2} + \frac{d\beta^2}{dx^2}} = \pm \frac{ds}{dx},$$ which is the proposition.

COR. Since $\frac{d\rho}{dx} = \pm \frac{ds}{dx}$, $\frac{d\rho}{dx} \pm \frac{ds}{dx} = 0$ Now, this is the differential coefficient of $\rho \pm s = C$, where C is a constant quantity.



Take $MNR = s$, and $RO = \rho$, then $MNR + RO = C$. Let a string having its extremity fixed at M be wound round the curve from M to R, and when unwound, let a pencil at O describe the curve OPQ. Then, because the curve OPQ is thus described, it is called the involute of MNR, and MNR, on the other hand, is for the same reason called the evolute of OPQ.

EXAMPLE (1.) Find the direction of curvature in the cubical parabola whose equation is $y = a^{\frac{1}{3}} x^{\frac{2}{3}}$.

Here $\frac{dy}{dx} = \frac{1}{3}\frac{a^{\frac{1}{3}}}{x^{\frac{2}{3}}}$, and $\frac{d^2y}{dx^2} = -\frac{2}{9}\frac{a^{\frac{1}{3}}}{x^{\frac{5}{3}}}$.

Therefore if x be positive, $\frac{d^2y}{dx^2}$ is negative, and the curve has its concavity towards the axis of x.

Ex. (2.) Find the direction of curvature in a hyperbola whose polar equation is $r^2 = \frac{a^2(e^2-1)}{e^2\cos.^2\theta - 1}$;

Therefore $p = \frac{ab}{\sqrt{r^2 - a^2 + b^2}}$, and $\frac{dp}{dr} = -\frac{abr}{(r^2-a^2+b^2)^{\frac{3}{2}}}$. Hence the curve is convex to the pole.

Ex. (3.) Find the radius of curvature at any point in the parabola whose equation is $y^2 = 4mx$.

$\therefore \frac{dy}{dx} = \frac{m^{\frac{1}{2}}}{x^{\frac{1}{2}}}$ and $\frac{d^2y}{dx^2} = -\frac{m^{\frac{1}{2}}}{2x^{\frac{3}{2}}}$.

$\frac{dy^2}{dx^2} = \frac{m}{x}$. But $\rho = \frac{\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}} = \frac{2(m+x)^{\frac{3}{2}}}{m^{\frac{1}{2}}}$ and when $x = 0$, $\rho = 2m$.

Ex. (4.) Find the radius of curvature in the rectangular hyperbola, whose equation referred to its asymptotes is $xy = m^2$.

$\therefore \frac{dy}{dx} = -\frac{m^2}{x^2}$ and $\frac{dy^2}{dx^2} = \frac{m^4}{x^4} = \frac{y^2}{x^2}$, also $\frac{d^2y}{dx^2} = \frac{2m^2}{x^3}$;

$\therefore \rho = \frac{(x^2+y^2)^{\frac{3}{2}}}{-2m^2}$.

Ex. (5.) Find the radius of curvature in the cycloid, whose equation is $y = a \text{ vers.}^{-1} \frac{x}{a} + \sqrt{2ax - x^2}$.

$$\frac{dy}{dx} = \left(\frac{2a - y}{y}\right)^{\frac{1}{2}}; \text{ and } 1 + \frac{dy^2}{dx^2} = \frac{2a}{y}, \text{ also } \frac{d^2y}{dx^2} = -\frac{a}{y^2};$$

$$\therefore \varrho = \frac{\left(\frac{2a}{y}\right)^{\frac{3}{2}}}{\frac{a}{y^2}} = 2\,(2ay)^{\frac{1}{2}}.$$

Ex. (6.) Find the radius of curvature in the cardioid, whose polar equation is $r = a\,(1 - \cos.\ \theta.)$

$$\frac{dr}{d\theta} = a \sin.\ \theta, \text{ and } \frac{d^2r}{d\theta^2} = a \cos.\ \theta; \therefore \varrho = \frac{\left(r^2 + \frac{dr^2}{d\theta^2}\right)^{\frac{3}{2}}}{r^2 + 2\frac{dr^2}{d\theta^2} - r\frac{d^2r}{d\theta^2}}$$

$$= \frac{(r^2 + a^2 \sin.^2 \theta)^{\frac{3}{2}}}{r^2 + 2a^2 \sin.^2 \theta - ar \cos.\ \theta} = \frac{(2ar)^{\frac{3}{2}}}{3ar} = \frac{(8ar)^{\frac{1}{2}}}{3}$$

Ex. (7.) Find the radius of curvature in the trisectrix, whose polar equation is $r = a\,(2 \cos.\ \theta \pm 1.)$

$$\frac{dr}{d\theta} = -2a \sin.\ \theta, \quad \frac{d^2r}{d\theta^2} = -2a \cos.\ \theta;$$

$$\therefore \varrho = \frac{(a^2 (2 \cos.\ \theta \pm 1)^2 + 4a^2 \sin^2 \theta)^{\frac{3}{2}}}{a^2 (2 \cos.\ \theta \pm 1)^2 + 8a^2 \sin.^2 \theta + 2a^2 \cos.\ \theta\,(2 \cos.\ \theta \pm 1)}$$

$$= \frac{a\,(5 \pm 4 \cos.\ \theta)^{\frac{3}{2}}}{3 \quad 3 \pm 2 \cos.\ \theta}$$

Ex. (8.) Find the evolute in the common parabola.

$$y^2 = 4ax; \quad \therefore p = \frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}}, \quad q = -\frac{a^{\frac{1}{2}}}{2x^{\frac{3}{2}}}.$$

$$\alpha = x - \frac{1+p^2}{q}p = 3x + 2a;\ \therefore x = \frac{\alpha - 2a}{3}$$

$$\beta = y + \frac{1+p^2}{q} = -\frac{y^3}{4a^2};\ \therefore y = (-4a^2\beta)^{\frac{1}{3}}.$$

Substituting these values of x and y in the equation to the parabola, we have $27\ a\beta^2 = 4\ (\alpha - 2a)^3$ the equation to the evolute.

Ex. (9.) The equation to the tractrix is $a + \sqrt{a^2 - y^2} = y\,e^{\frac{x+\sqrt{a^2-y^2}}{a}}$: find the equation to its evolute.

$$\log.\,(a + (a^2 - y^2)^{\frac{1}{2}}) = \log.\,y + \frac{x + (a^2 - y^2)^{\frac{1}{2}}}{a};$$

$$\therefore -\frac{yp}{a + (a^2 - y^2)^{\frac{1}{2}}} = (a^2 - y^2)^{\frac{1}{2}}\frac{p}{y} + \frac{(a^2 - y^2)^{\frac{1}{2}} - yp}{a};$$

$$\therefore p = -\frac{y}{(a^2 - y^2)^{\frac{1}{2}}} \text{ and } q = \frac{a^2 y}{(a^2 - y^2)^2};$$

$$\therefore \alpha = x + (a^2 - y^2)^{\frac{1}{2}} \text{ and } \beta = \frac{a^2}{y}\ \therefore y = \frac{a^2}{\beta},$$

and $x = \alpha - \frac{a}{\beta}(\beta^2 - a^2)^{\frac{1}{2}}$. Substituting these values for x and y in the equation, we have $e^{\frac{\alpha}{a}} = \frac{\beta \pm \sqrt{\beta^2 - a^2}}{a}$; $\therefore \alpha = a \log.\frac{\beta + \sqrt{\beta^2 - a^2}}{a}$, which is the equation to the evolute.

Ex. (10). The equation to a curve is $r = a^{\alpha - \theta}$: find the equation to its evolute.

$$\log.\,r = (\alpha - \theta)\log.\,a,\ \therefore \frac{dr}{d\theta} = -r\log.\,a, \text{ and } \frac{d^2r}{d\theta^2} = r\log.^2 a;$$

$$\therefore p = \frac{r^2}{\sqrt{r^2 + \frac{dr^2}{d\theta^2}}} = \frac{r}{(1 + \log.^2 a)^{\frac{1}{2}}};\ \therefore \frac{dp}{dr} = \frac{1}{(1 + \log.^2 a)^{\frac{1}{2}}}, \text{ and}$$

$\frac{dr}{dp} = (1 + \log.^2 a)^{\frac{1}{2}}$, $\frac{dr^2}{dp^2} = 1 + \log.^2 a$. Hence $\rho = r \log. a$, and $p' =$

$\frac{r \log. a}{(1 + \log.^2 a)^{\frac{1}{2}}}$, $\therefore p' = \frac{r'}{(1 + \log.^2 a)^{\frac{1}{2}}}$, which is the equation to the evolute.

EXAMPLES FOR PRACTICE.

(1.) The equation to a curve is $(y - b)^2 = x(x - a)^2$, prove that it is convex to the axis of x.

(2.) The equation to a curve is $r = a(\cos. \theta - \sin. \theta)$, prove that it is concave to the pole.

(3.) Prove that the curve whose equation is $r = a\theta^n$, is concave to the pole.

(4.) The equation to a conic section, whose latus rectum is m, is $y = \sqrt{mx + nx^2}$, prove that $\rho = -\frac{4}{m^2}\left(\frac{1}{4}m^2 + (n + 1)y^2\right)^{\frac{3}{2}}$.

(5.) The equation to the catenary is $y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)$, prove that $\rho = -\frac{y^2}{c}$.

(6.) The equation to an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, prove that $\rho = \frac{(a^2 - e^2 x^2)^{\frac{3}{2}}}{ab}$

(7.) The equation to a curve is $r = a(\cos. \theta - \sin. \theta)$, prove that $\varrho = \frac{1}{2} a \sqrt{2}$.

(8.) The equation to the lituus is $r^2 = \frac{a^2}{\theta}$, prove that, $\varrho = \frac{r}{2a^2} \frac{(4a^4 + r^4)^{\frac{3}{2}}}{4a^4 - r^4}$.

(9.) The equation to an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, prove that $(a\alpha)^{\frac{2}{3}} + (b\beta)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$ is the equation to its evolute.

(10. The equation to the hypocycloid is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, prove that the equation to its evolute is $(\alpha + \beta)^{\frac{2}{3}} + (\alpha - \beta)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$.

(11.) The equation to a curve is $\sqrt{r^2 - a^2} = a\theta + a \sec.^{-1} \frac{r}{a}$, prove that its evolute is a circle whose radius is a.

(12.) The equation to the epicycloid is $p^2 = \frac{c^2(r^2 - a^2)}{c^2 - a^2}$, prove that its evolute is an epicycloid whose equation is $p'^2 = \frac{c^2\left(r'^2 - \frac{a^4}{c^2}\right)}{c^2 - a^2}$.

CHAPTER XIV.

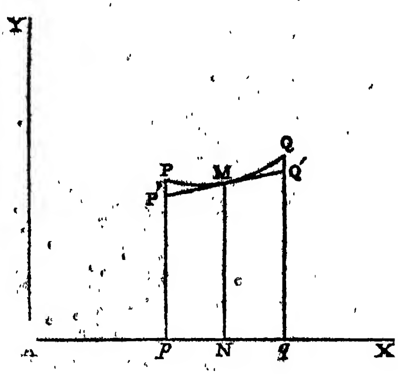
SINGULAR POINTS IN CURVES, TRACING CURVES FROM THEIR EQUATIONS.

(124) Points where curves undergo any particular changes are called *singular points.*

POINTS OF INFLEXION, OR CONTRARY FLEXURE.

(125.) A point where a curve changes from being convex to the axis to concave, or *vice versa*, is called a *point of inflexion* or of *contrary flexure.*

(126.) To find the points of inflexion of a curve referred to rectangular co-ordinates.



It appears from (115) that

$$QQ' = \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\frac{h^3}{1.2.3} + \frac{d^4y}{dx^4}\frac{h^4}{1.2.3.4} + \dots$$

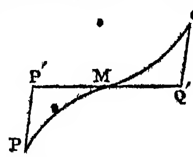
$$\text{and } P P' = \frac{d^2y}{dx^2}\frac{h^2}{1.2} - \frac{d^3y}{dx^3}\frac{h^3}{1.2.3} + \frac{d^4y}{dx^4}\frac{h^4}{1.2.3.4} - \ldots$$

But when h is taken very small, the first terms of these series are greater than the sum of all those that follow them; $\therefore$ Q Q′ and P P′ have always the same sign, and therefore the curve lies wholly on the same side of the tangent P′ M Q′; and therefore there cannot be a point of inflexion unless $\frac{d^2y}{dx^2} = 0$. In which case,

$$Q Q' = \frac{d^3y}{dx^3}\frac{h^3}{1.2.3} + \frac{d^4y}{dx^4}\frac{h^4}{1.2.3.4} + \frac{d^5y}{dx^5}\frac{h^5}{1.2.3.4.5} + \ldots$$

$$\text{and } P P' = -\frac{d^3y}{dx^3}\frac{h^3}{1.2.3} + \frac{d^4y}{dx^4}\frac{h^4}{1.2.3.4} - \frac{d^5y}{dx^5}\frac{h^5}{1.2.3.4.5} + \ldots$$

Now, when h is taken very small, the signs of QQ′ and PP′ are the same as those of the first terms of the above series; $\therefore$ the curve cuts the straight line P′MQ′ at M, or has a point of inflexion as in the adjoining figure.



(127.) Again, if the same value of x which renders $\frac{d^2y}{dx^2} = 0$, render $\frac{d^3y}{dx^3} = 0$, there cannot be a point of inflexion unless it also render $\frac{d^4y}{dx^4} = 0$; and if it render $\frac{d^5y}{dx^5} = 0$, it must also render $\frac{d^6y}{dx^6} = 0$, or it is necessary that the last differential coefficient that vanishes be of an even order if there be a point of inflexion.

(128.) The demonstration given above goes upon the supposition that if $y = f(x)$ be the equation to a curve, and x be increased or diminished by a very small quantity h, the corresponding values of y may be found in series of ascending powers of h. But if a particular value of x render $\frac{d^2y}{dx^2} = \infty$, it is clear that this is a case where Taylor's Theorem fails, and that consequently nothing can be inferred from the above demonstration.

It may be here remarked that $\frac{d^2y}{dx^2} = 0$ in general implies that $\frac{d^2y}{dx^2}$ changes its sign when passing through the point where it is equal to zero; but it may also change its sign when passing through infinity. Therefore, when a particular value of x renders $\frac{d^2y}{dx^2} = \infty$, if that value when increased and diminished successively by a very small quantity h, causes $\frac{d^2y}{dx^2}$ to change its sign, it implies that there is a point of inflexion at the point whose absciss is the particular value of x which renders $\frac{d^2y}{dx^2} = \infty$.

For example, let $y = \frac{b^2}{2(x-a)}$, then $\frac{dy}{dx} = -\frac{b^2}{2(x-a)^2}$, and $\frac{d^2y}{dx^2} = \frac{b^2}{(x-a)^3}$.

$$\text{Let } x = a + h, \text{ then } \frac{d^2y}{dx^2} = +\frac{b^2}{h^3}.$$

$$x = a, \quad \ldots \quad \frac{d^2y}{dx^2} = \infty.$$

$$x = a - h, \ldots \quad \frac{d^2y}{dx^2} = -\frac{b^2}{h^3}.$$

(129.) It appears, therefore, that when $\frac{d^2y}{dx^2} = 0$, or $\frac{d^2y}{dx^2} = \infty$, there may be a point of inflexion; and when a particular value of x causes the second differential coefficient to fulfil any one of these two conditions, it is merely necessary to increase and diminish this value of x by a very small quantity h; and if $\frac{d^2y}{dx^2}$ change its sign, the point of inflexion is determined.

EXAMPLE (1). Find whether the curve whose equation is $y = b + 2\,(x-a)^3$ has a point of inflexion. $\frac{dy}{dx} = 6\,(x-a)^2$; $\frac{d^2y}{dx^2} = 12\,(x-a)$; and if there be a point of inflexion, $\frac{d^2y}{dx^2} = 0$, or $\frac{d^2y}{dx^2} = \infty$. But it is obvious that in this case $\frac{d^2y}{dx^2} = 0$ when $x = a$, and $\frac{d^2y}{dx^2} = +\,12\,h$, when $x = a + h$, and $\frac{d^2y}{dx^2} = -\,12\,h$ when $x = a - h$; $\therefore$ the curve has a point of inflexion at the point where $x = a$ and $y = b$.

Ex. (2). The equation to the witch of Agnesi is $xy = 2\,a^2(2\,ax - x^2)^{\frac{1}{2}}$: find whether it has a point of inflexion.

$\frac{d^2y}{dx^2} = \frac{2\,a^2\,(3\,a - 2\,x)}{x\,(2\,ax - x^2)^{\frac{3}{2}}} = 0$, $\therefore\ x = \frac{3\,a}{2}$, and $y = \pm\,\frac{2\,a}{3^{\frac{1}{2}}}$; and when $\frac{3\,a}{2} + h$ and $\frac{3\,a}{2} - h$ are substituted for x, $\frac{d^2y}{dx^2}$ changes its sign; therefore there are two points of inflexion corresponding to these values of x and y.

Ex. (3.) The equations to the companion of the cycloid are $x = a\theta$, and $y = a\,(1 + \cos.\,\theta)$: find whether it has a point of inflexion.

Since $\theta = \frac{x}{a}$, $y = a\left(1 + \cos.\frac{x}{a}\right)$; $\therefore\ \frac{d^2y}{dx^2} = -\,\frac{1}{a}\cos.\frac{x}{a} = 0$ when $x = \frac{1}{2}\,\pi\,a$, and when $\frac{1}{2}\,\pi\,a + h$, and $\frac{1}{2}\,\pi\,a - h$ are substituted for x, we have $\frac{d^2y}{dx^2} = +\,\frac{1}{a}\sin.\frac{h}{a}$, and $\frac{d^2y}{dx^2} = -\,\frac{1}{a}\sin.\frac{h}{a}$; $\therefore$ there is a point of inflexion when $x = \frac{1}{2}\,\pi\,a$, and $y = a$.

Ex. (4.) Find whether the curve whose equation is $x = (y - b)^{\frac{2}{3}}$ has a point of inflexion.

$$y = b \pm x^{\frac{3}{2}};\ \therefore \frac{d^2y}{dx^2} = \pm \frac{3}{4\,x^{\frac{1}{2}}} = \pm \infty, \text{ when } x = 0.$$ Let $0 + h$, and $0 - h$, be successively substituted for x, and we shall have

$$\frac{d^2y}{dx^2} = \pm \frac{3}{4\,h^{\frac{1}{2}}}, \text{ and } \frac{d^2y}{dx^2} = \pm \frac{3}{4\,(-h)^{\frac{1}{2}}},$$ which last expression is imaginary, and therefore the curve has no point of inflexion.

(130.) To find the points of inflexion of a curve referred to polar co-ordinates.

It appears from (116) that when $\frac{dp}{dr}$ is positive, the curve is concave to the pole; and when $\frac{dp}{dr}$ is negative, it is convex; $\therefore \frac{dp}{dr}$ must change it sign in passing through zero or infinity; and hence, when there is a point of inflexion $\frac{dp}{dr} = 0$, or $\frac{dp}{dr} = \infty$.

Example (1.) The equation to a curve is $r = a\,(1 + \cos.\ \theta)$, find whether it has a point of inflexion.

$$\frac{dr}{d\theta} = -\,a \sin.\ \theta;\ \therefore p = \frac{r^2}{\sqrt{r^2 + a^2 \sin.^2 \theta}} = \frac{r^{\frac{3}{2}}}{\sqrt{2\,a}};\ \therefore \frac{dp}{dr} = \frac{3\,r^{\frac{1}{2}}}{2\sqrt{2\,a}}$$
$= 0$; $\therefore r = 0$; that is, there is a point of inflexion at the pole.

Ex. (2.) The equation to a curve is $r = \frac{a\theta^2}{\theta^2 - 1}$, find whether it has a point of inflexion.

$$\frac{dr}{d\theta} = -\frac{2\,a\theta}{(\theta^2 - 1)^2}, \text{ and } \theta = \frac{r^{\frac{1}{2}}}{(r - a)^{\frac{1}{2}}};\ \therefore \frac{dr}{d\theta} = -\,\frac{2\,r^{\frac{1}{2}}\,(r - a)^{\frac{3}{2}}}{a}.$$

But $p = \dfrac{r^2}{\sqrt{r^2 + \dfrac{dr^2}{d\theta^2}}} = \dfrac{ar^2}{\sqrt{a^2r^2 + 4r(r-a)^3}}$, and

$$\frac{dp}{dr} = -\frac{a^2 r^{\frac{1}{2}}(6r^2 - 13ar + 6a^2)}{(a^2r + 4(r-a)^3)^{\frac{3}{2}}} = 0, \text{ when there is point of inflexion;}$$

$\therefore 6r^2 - 13ar + 6a^2$, and $r = \dfrac{3a}{2}$, or $= \dfrac{2a}{3}$.

The first value of r gives $\theta = \pm\sqrt{3}$, and the second $\theta = \sqrt{-2}$. But as this second value of θ is imaginary, there is only one point of inflexion in this curve at the distance of $\dfrac{3a}{2}$ from the pole.

POINTS OF REFLEXION OR CUSPS.

(131.) A point in which a curve stops in its course and turns back, is called a point of reflexion or cusp.

When the two branches of the curve have their convexities turned in opposite directions, as in figure (1), the curve is said to have a point

Fig 1.

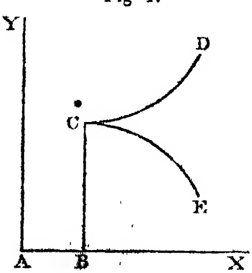
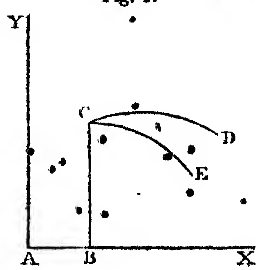


Fig. 2.



of reflexion of the first species; but if their convexities be turned in the same direction as in fig. (2), the point of reflexion is of the second species.

(132.) To find the points of reflexion or cusps of a given curve.

A cusp or point of reflexion arises in this way. Any value of x less than A B renders the corresponding values of y imaginary, and this implies that $\frac{d^2y}{dx^2}$ contains a surd. Any value of x a little greater than A B must give two values to $\frac{d^2y}{dx^2}$. If these values have opposite signs, the two branches of the curve must have their convexities turned towards each other; but if they have the same sign, the branches must have their convexities turned either both to or both from the axis of x (115.)

Example (1.) The equation to the semicubical parabola is $ay^2 = x^3$, find whether it has a point of reflexion.

$$y = \pm \frac{x^{\frac{3}{2}}}{a^{\frac{1}{2}}}, \ \frac{dy}{dx} = \pm \frac{3\,x^{\frac{1}{2}}}{2\,a^{\frac{1}{2}}}, \ \frac{d^2y}{dx^2} = \pm \frac{3}{4\,a^{\frac{1}{2}}\,x^{\frac{1}{2}}}.$$

If $x = 0$, $y = 0$, and if $0 - h$ be substituted for x, y is imaginary; therefore no part of the curve corresponds to negative abscissæ.

Substitute $0 + h$ for x in the values of $\frac{d^2y}{dx^2}$, and they become $+\frac{3}{4\,a^{\frac{1}{2}}\,h^{\frac{1}{2}}}$, and $-\frac{3}{4\,a^{\frac{1}{2}}\,h^{\frac{1}{2}}}$. Therefore the curve has a cusp at the origin of the *first* species.

Ex. (2.) The equation to a curve is $x\,(y - 1)^2 = (2 - x)^3$, find whether it has a cusp.

$$y = \pm \frac{(2-x)^{\frac{3}{2}}}{x^{\frac{1}{2}}} + 1, \ \frac{dy}{dx} = \mp \frac{(x+1)\,(2-x)^{\frac{1}{2}}}{x^{\frac{3}{2}}}, \ \frac{d^2y}{dx^2} = \pm \frac{3}{x^{\frac{5}{2}}\,(2-x)^{\frac{1}{2}}}.$$

When $x = 2$, $y = 1$, substitute $2 + h$ for x, and the value of y is

imaginary. Substitute $2 - h$ for x in the values of $\frac{d^2y}{dx^2}$, and they become $+\frac{3}{(2-h)^{\frac{5}{3}}h^{\frac{1}{3}}}$, and $-\frac{3}{(2-h)^{\frac{5}{3}}h^{\frac{1}{3}}}$. Therefore the curve has a cusp of the *first* species at the point $x = 2$, and $y = 1$.

Ex. (3.) The equation to a curve is $(y - ax^2)^2 = b^2 x^5$, find whether it has a point of reflexion.

$$y = ax^2 \pm bx^{\frac{5}{2}},\ \frac{dy}{dx} = 2\,ax \pm \frac{5}{2}\,bx^{\frac{3}{2}},\ \frac{d^2y}{dx^2} = 2\,a \pm \frac{15}{4}\,bx^{\frac{1}{2}}.$$

When $x = 0$, $y = 0$, substitute $0 - h$ for x, and y is imaginary. Substitute $0 + h$ for x in the values of $\frac{d^2y}{dx^2}$, and they become $2\,a + \frac{15\,bh^{\frac{1}{2}}}{4}$, and $2\,a - \frac{15\,bh^{\frac{1}{2}}}{4}$; and if h be very small, $2\,a > \frac{15\,bh^{\frac{1}{2}}}{4}$.

$\therefore$ both these values of $\frac{d^2y}{dx^2}$ have the same sign, and the cusp is of the *second* species.

Ex. (4.) The equation to the cissoid of Diocles is $(2\,a - x)\,y^2 = x^3$, find whether it has a cusp, and the directions of the tangents.

$$y = \pm\frac{x^{\frac{3}{2}}}{(2\,a - x)^{\frac{1}{2}}},\ \frac{dy}{dx} = \pm\frac{(3\,a - x)\,x^{\frac{1}{2}}}{(2\,a - x)^{\frac{3}{2}}},\ \frac{d^2y}{dx^2} = \pm\frac{3\,a^2}{x^{\frac{1}{2}}\,(2\,a - x)^{\frac{5}{2}}}.$$

When $x = 0$, $y = 0$, and when $0 - h$ is substituted for x, y is imaginary. Let $0 + h$ be substituted for x in the values of $\frac{d^2y}{dx^2}$, and they become $+\frac{3\,a^2}{h^{\frac{1}{2}}\,(2\,a - h)^{\frac{5}{2}}}$, and $-\frac{3\,a^2}{h^{\frac{1}{2}}\,(2\,a - h)^{\frac{5}{2}}}$. Therefore the curve has a cusp of the first species at the origin, and since $\frac{dy}{dx} = \pm 0$, the

two branches have the same tangent which coincides with the axis of x.

MULTIPLE POINTS.

(133.) A point in which several branches of a curve intersect, is called a multiple point. It is called a double, triple, &c. point, according as it is common to two, three, &c. branches of the curve.

Fig. 1.

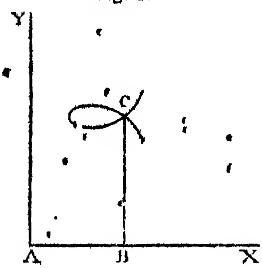


Fig. 2.

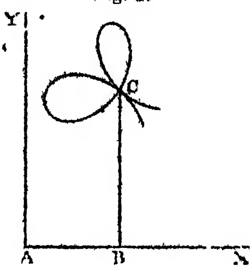


Fig. (1.) is a double, and fig. (2.) a triple point.

(134.) To find the multiple points of a given curve.

Let $F(x, y) = 0$ be the equation to a curve freed of surds, and if, when a particular value A B is given to x, the ordinate y have only one value; but the tangent $\frac{dy}{dx}$ have two or more values, then it is obvious that the point determined by this value of x must be a multiple point.

Since $F(x, y) = 0$, we have by differentiation $M + N\frac{dy}{dx} = 0$, or $\frac{dy}{dx} = -\frac{M}{N}$. Here $\frac{dy}{dx}$ must have two or more values, and as the equation $F(x, y) = 0$ was cleared of surds by hypothesis, and as no surd can be introduced into an equation by differentiation, $\frac{M}{N}$ must be

of the form $\frac{0}{0}$. Let $\frac{dy}{dx}$ have two values α and β, then $M + N\alpha = 0$, and $M + N\beta = 0$; $\therefore N(\alpha - \beta) = 0$. But, since the two values of $\frac{dy}{dx}$ are unequal, N must be equal to zero, and hence M must also be equal to zero. Wherefore, when a particular value is given to x, and y has only one value, it is necessary that M and N be each equal to nothing, in order that the point may be a double point.

This demonstration obviously holds, whatever be the number of branches which intersect in the same point, if each have a *separate* tangent; but it does not apply if two or more branches have a *common* tangent. In this case, since the tangent has contact of the first order with each curve—Ex. (1.) (118.)—these curves must be osculates. Let them have contact of the nth order, then, if we differentiate $M + N\frac{dy}{dx} = 0$, n times, we will have $R + N\frac{d^{n+1}y}{dx^{n+1}} = 0$; $\therefore \frac{d^{n+1}y}{dx^{n+1}} = -\frac{R}{N}$; and in order that this may have two valves, $\frac{R}{N}$ must be $= \frac{0}{0}$; and it may be proved, as in the first case, that $R = 0$, and $N = 0$. Hence $M = 0$; $\therefore \frac{dy}{dx} = -\frac{M}{N} = \frac{0}{0}$ as before.

It may be proper to state here, that $\frac{dy}{dx} = \frac{0}{0}$ does not in every case prove the existence of a multiple point. It merely shews that such a point may exist, and by examining the curve in the neighbourhood of that point, we can easily ascertain whether it be a multiple point or not.

EXAMPLE (1.) The equation to a curve is $(y-2)^2 = (x-1)^2 x$: find whether it has a multiple point.

Since $(y-2)^2 = (x-1)^2 x$, when $x = 1$, $y = 2$, and $\frac{dy}{dx} =$

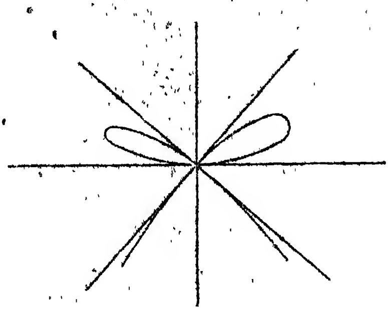
$$\frac{3x^2 - 4x + 1}{2(y-2)} = \frac{0}{0} = \frac{3x-2}{\frac{dy}{dx}};\ \therefore \frac{dy^2}{dx^2} = 3x - 2 = 1,\ \therefore \frac{dy}{dx} = \pm 1.$$

Therefore two branches of the curve intersect at the point $x=1$, $y=2$, and are inclined to a parallel to the axis of x at angles of 45° and 135°.

Ex. (2.) The equation to a curve is $x^4 - ayx^2 + ay^3 = 0$: find whether it has a multiple point.

$$\frac{dy}{dx} = \frac{2\,ayx - 4\,x^3}{3\,ay^2 - ax^2} = \frac{0}{0} = \frac{ax\dfrac{dy}{dx} + ay - 6\,x^2}{3\,ay\dfrac{dy}{dx} - ax} = \frac{0}{0};\ \therefore$$

$$\frac{2\,a\dfrac{dy}{dx} - 12\,x}{3\,a\left(\dfrac{dy}{dx}\right)^2 - a} = \frac{dy}{dx};\ \therefore 3a\left(\frac{dy}{dx}\right)^3$$



$- 3a\dfrac{dy}{dx} + 12x = 0$; and when

$x = 0,\ y = 0,\ \therefore \left(\dfrac{dy}{dx}\right)^3 - \dfrac{dy}{dx}$

$= 0$: $\therefore \dfrac{dy}{dx} = \pm 1$ and $= 0$. There is therefore a triple point at the origin of co-ordinates, as in the figure.

Ex. (3.) The equation to a curve is $(x^2 + y^2)^3 = 4\,a^2x^2y^2$: find whether it has a multiple point.

$$\frac{dy}{dx} = -\frac{4\,a^2xy^2 - 3\,(x^2+y^2)^2x}{4\,a^2x^2y - 3\,y\,(x^2+y^2)^2} = \frac{0}{0};$$

$$\text{Therefore } 4\,a^2xy^2 - 3\,(x^2+y^2)^2x = 0;$$

$$4a^2x^2y - 3\,y\,(x^2+y^2)^2 = 0;$$

$$(x^2+y^2)^3 - 4\,a^2x^2y^2 = 0.$$

Therefore it is necessary that $x = 0$, and $y = 0$, to satisfy these three conditions; and the values of $\dfrac{dy}{dx}$, found in the usual way, shew that

there is a quadruple point at the origin, and that the axes of co-ordinates are tangents to the different branches of the curve.

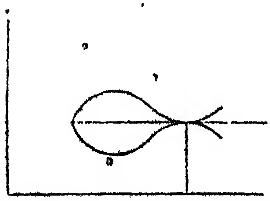
Let the origin of co-ordinates be taken as the pole, then $x = r \cos. \theta$, and $y = r \sin \theta$; $\therefore (x^2 + y^2)^3 = r^6 = 4 a^2 r^4 \sin.^2 \theta \cos.^2 \theta$; $\therefore r = 2 a \sin. \theta \cos. \theta = a \sin. 2 \theta$, which is the polar equation to the curve, from which its form may be easily seen.

Ex. (4.) The equation to a curve is $(y - c)^2 = (x - a)^4 (x - b)$, $a > b$: find whether it has a multiple point.

$$\frac{dy}{dx} = \frac{(x-a)^3 (5x - a - 4b)}{2(y-c)} = \frac{0}{0}, \quad \therefore \frac{x(-a)^2 (10x - 4a - 6b)}{\frac{dy}{dx}} = \frac{dy}{dx};$$

$$\therefore \quad \frac{dy}{dx} = 0, \text{ when } x = a.$$

Therefore the curve has only one tangent at the point $x = a$, $y = c$, which is parallel to the axis of x. By proceeding in the same manner we find that $\frac{d^2y}{dx^2}$ has both a positive and negative value; therefore the curve has a double point, as in the figure; and the two branches have contact of the first order.



ISOLATED OR CONJUGATE POINTS.

(135.) When a particular value of the abscissa gives a real value to the ordinate; but the same value of the abscissa increased or diminished by a small quantity renders the ordinate imaginary—the point thus determined, being detached from the rest of the curve, is called an isolated or conjugate point.

(136.) To find the conjugate points of a given curve.

Let the equation to the curve be $y = f(x)$, and let x become equal to $x \pm h$, then

$$f(x \pm h) = y \pm \frac{dy}{dx} h + \frac{d^2y}{dx^2} \frac{h^2}{1.2} \pm \frac{d^3y}{dx^3} \frac{h^3}{1.2.3} + \frac{d^4y}{dx^4} \frac{h^4}{1.2.3.4} \pm \&c.$$

When $x = a$ let $y = b$, and the differential co-efficients be represented by $\left(\frac{dy}{dx}\right)$, $\left(\frac{d^2y}{dx^2}\right)$, $\left(\frac{d^3y}{dx^3}\right)$, &c. then

$$f(a \pm h) = b \pm \left(\frac{dy}{dx}\right)\frac{h}{1} + \left(\frac{d^2y}{dx^2}\right)\frac{h^2}{1.2} + \left(\frac{d^3y}{dx^3}\right)\frac{h^3}{1.2.3} + \&c.$$

But $f(a \pm h)$ is imaginary by hypothesis, therefore one at least of the differential co-efficients must be imaginary.

Hence conversely, when $y = f(x)$ and x becomes equal to $x \pm h$, if one of the differential co-efficients in the development become imaginary, when a particular value is given to x, there may be a conjugate point.

(137.) When there is a conjugate point $\frac{dy}{dx} = \frac{0}{0}$. For the equation to the curve may be put under the form $F(x, y) = 0$; $\therefore M + N\frac{dy}{dx} = 0$.

By differentiating a second time, we have $N\frac{d^2y}{dx^2} + \frac{dy}{dx}\frac{dN}{dx} + \frac{dM}{dx} = 0$

$= N\frac{d^2y}{dx^2} + P = 0$; $\therefore N\frac{d^ny}{dx^n} + R = 0$, and $\frac{d^ny}{dx^n} = -\frac{R}{N}$. But one of the differential co-efficients is imaginary; $\therefore$ it must have at least two values, since it contains a surd. Let that co-efficient be $\frac{d^ny}{dx^n}$, then it may be proved, as in (131), that $N = 0$, $R = 0$, and also $M = 0$. But $\frac{dy}{dx} = -\frac{M}{N}$; $\therefore \frac{dy}{dx} = \frac{0}{0}$.

Example (1.) Let the equation to a curve be $y^2 = \frac{x^2}{a}(x - b)$: find whether it has a conjugate point.

$\frac{dy}{dx} = \pm \frac{3x - 2b}{a^2 (x - b)^{\frac{1}{2}}}$, and when $x = 0$, $y = 0$, and the corresponding values of $\frac{dy}{dx}$ are $-\frac{2b}{a^2\sqrt{-b}}$ and $+\frac{2b}{a^2\sqrt{-b}}$, which are both imaginary; $\therefore$ the origin of co-ordinates is a conjugate point.

Ex. (2.) Let the equation to a curve be $(y - 3)^2 = (x + 2)^2 (x + 1)$, find whether it has a conjugate point.

$y = \pm (x + 2)(x + 1)^{\frac{1}{2}} + 3$, $\frac{dy}{dx} = \pm \frac{3x + 4}{2(x + 1)^{\frac{1}{2}}}$. Let $x = -2$, then $y = 3$, and the values of $\frac{dy}{dx}$ are $+\sqrt{-1}$ and $-\sqrt{-1}$, which are imaginary, therefore the ordinates contiguous to the point -2 and 3 have no existence; it is therefore an isolated or conjugate point.

Ex. (3.) Let $(y + a)^2 = (x - b)^2 (x - 2b)$ be the equation to a curve, find whether it has a conjugate point.

$$\frac{dy}{dx} = \frac{(x - b)(3x - 5b)}{2(y + a)}. \quad \text{Let } x = b, \text{ then } y = -a,$$

$$\text{and } \frac{dy}{dx} = \frac{0}{0} = \frac{3x - 4b}{\frac{dy}{dx}}; \quad \therefore \left(\frac{dy}{dx}\right)^2 = 3x - 4b = -b;$$

$\therefore \frac{dy}{dx} = \pm\sqrt{-b}$, which values of $\frac{dy}{dx}$ are imaginary; therefore the point $(b, -a)$ is a conjugate point.

Otherwise, let $b \pm k$ be substituted for x in the equation to the curve, k being a very small quantity, then the values of y are imaginary; therefore the point $(b, -a)$ is detached from the rest of the curve, and is therefore a conjugate point.

POINTS OF MAXIMUM OR MINIMUM CURVATURE.

(138.) It appears from (119) that $\varrho = \frac{(1+p^2)^{\frac{3}{2}}}{-q}$, when x is the independent variable. If therefore such a value be assigned to x as shall render $\frac{d\varrho}{dx} = 0$, or $\frac{d\varrho}{dx} = \infty$, then, according as $\frac{d^2\varrho}{dx^2}$ is positive or negative, the corresponding point in the curve will have its radius of curvature a minimum or maximum.

It is obvious that when the radius of curvature is a maximum or minimum, the curvature itself will be a minimum or maximum.

EXAMPLE (1.) The equation to the Logarithmic Spiral is $y = a^x$, find its point of maximum curvature.

$$\frac{dy}{dx} = a^x \log. a, \; \frac{d^2y}{dx^2} = a^x (\log. a)^2; \therefore \varrho = \frac{(1 + a^{2x} (\log. a)^2)^{\frac{3}{2}}}{-a^x (\log. a)^2};$$

$$\frac{d\varrho}{dx} = \frac{(1 - 2a^{2x} (\log. a)^2)\, a^x (\log. a)^3 (1 + a^{2x} (\log. a)^2)^{\frac{1}{2}}}{a^{2x} (\log. a)^4} = 0;$$

$$\therefore a^{2x} = y^2 = \frac{1}{2 (\log. a)^2}; \; \therefore y = \frac{1}{2^{\frac{1}{2}} \log. a};$$ and because $\frac{d^2\varrho}{dx^2}$ is positive, ϱ is a minimum, and consequently the curvature a maximum.

Ex. (2.) The equation to a cycloid is $y = a \text{ vers.}^{-1} \frac{x}{a} + \sqrt{2ax - x^2}$, find its points of maximum and minimum curvature.

$$\frac{dy}{dx} = \left(\frac{2a - x}{x}\right)^{\frac{1}{2}}, \; \frac{d^2y}{dx^2} = -\frac{a}{(2a - x)^{\frac{1}{2}} x^{\frac{3}{2}}};$$

$$\therefore \rho = \frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}} = 2^{\frac{3}{2}} a^{\frac{1}{2}} (2a - x)^{\frac{1}{2}};$$ and when $x = 0$, and $x = 2a$, $y = 0$, and $y = \pi a$.

It hence appears, by the usual test, that the curvature is a minimum when $x = 0$ and $y = 0$, and a maximum when $x = 2a$, and $y = \pi a$.

THE TRACING OF CURVES FROM THEIR EQUATIONS.

(139.) To trace a curve referred to rectangular co-ordinates from its equation.

(1.) Let the equation to the curve be reduced, if possible, to the form $y = f(x)$.

(2.) Substitute all possible positive values, from 0 to ∞, for x, and observe which of them render $y = 0$, $y = \infty$, and y imaginary.

(3.) Substitute all possible negative values, from 0 to ∞, for x, and attend to the corresponding values of y as before.

(4.) Ascertain if the curve admits of asymptotes, and if it do draw them.

(5.) Find the values of $\frac{dy}{dx}$, and from them find at what angles the curve cuts the co-ordinate axes, and its maximum and minimum points.

(6.) Find the values of $\frac{d^2y}{dx^2}$, and ascertain from them when the curve is convex and concave to the axis of x.

(7.) Find the singular points of the curve by the rules already given.

Example (1.) Let $y^2 = 4ax$ be the equation to a curve, it is required to trace it.

$y = \pm 2 a^{\frac{1}{2}} x^{\frac{1}{2}}$.

$x = 0,\ y = 0$.

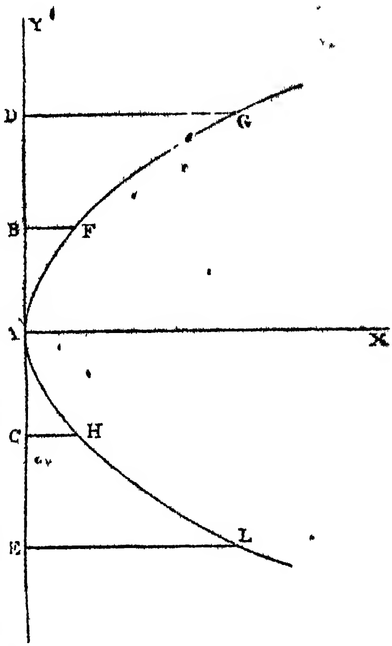
$x = a,\ y = \pm 2a$.

$x = 4a,\ y = \pm 4a$.

$x = \infty,\ y = \infty$.

$x = -a,\ y = \pm 2a\sqrt{-1}$.

$$\frac{dy}{dx} = \pm \frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}},\ \frac{d^2y}{dx^2} = \mp \frac{a^{\frac{1}{2}}}{2x^{\frac{3}{2}}}.$$



Therefore the curve passes through the origin A, and as y has two equal values with opposite signs for each positive value of x, the axis of x is a diameter of the curve; and since, when $x = \infty$, $y = \infty$, the curve has two infinite branches, A F G and A H L, one on each side of A X.

When x is negative, y is imaginary; therefore no part of the curve corresponds to negative abscissæ.

Again, since $\frac{dy}{dx} = \pm \frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}} = \infty$, when $x = 0$; $\therefore$ the curve cuts the axis of x at right angles at the origin; and since $\frac{d^2y}{dx^2} = \mp \frac{a^{\frac{1}{2}}}{2x^{\frac{3}{2}}}$ the value of $\frac{d^2y}{dx^2}$ is negative when y is positive, and positive when y is negative; $\therefore$ the curve is always concave to the axis of x.

Ex. (2.) The equation to a curve is $a^2 y = x^3$, it is required to trace it.

$$y = \frac{x^3}{a^2}.$$

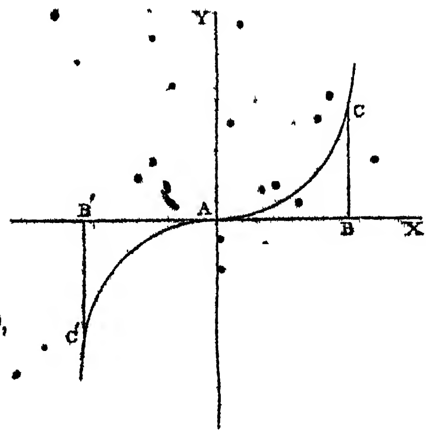
$x = 0, \quad y = 0.$

$x = a, \quad y = a.$

$x = \infty, y = \infty.$

$x = -a, \quad y = -a.$

$x = -\infty, y = -\infty.$

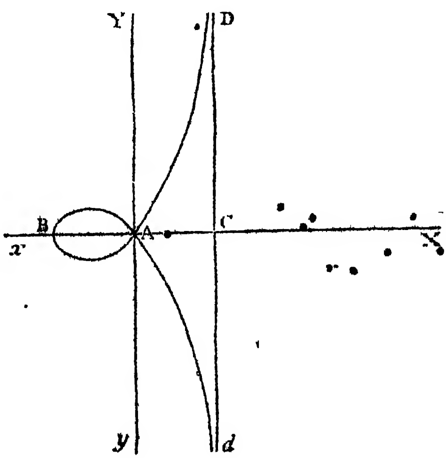


$\frac{dy}{dx} = \frac{3x^2}{a^2} = 0$, when $x = 0$,

and $\frac{d^2y}{dx^2} = \frac{6x}{a^2} = 0$;

and when $0 \pm h$ is substituted for x in the values of $\frac{d^2y}{dx^2}$, they become $\frac{6h}{a^2}$ and $-\frac{6h}{a^2}$; therefore the curve has a point of inflexion at the origin A, and has two infinite branches as in the figure; and since $\frac{dy}{dx} = 0$, the axis of x is a tangent to each branch at the point A.

Ex. (3.) The equation of a curve is $(a - x)y^2 = (a + x)x^2$, it is required to trace it.



$$y = \pm x\sqrt{\frac{a+x}{a-x}}$$

When $x = 0$, $y = 0$, $\therefore$ the curve passes through A, the origin of co-ordinates. When x is positive and less than a, y has two equal values with opposite signs, and when $x = a$, y is infinite; $\therefore$ the ordinate through C is an asymptote to the curve.

When x is negative and less than a, y has two equal values with opposite signs; and when $x = -a$, $y = 0$, $\therefore$ the curve passes through B; when $x > -a$, y is imaginary, and no part of the curve lies to the left of B.

$$\frac{dy}{dx} = \pm\frac{a^2 + ax - x^2}{a^2 - x^2}\sqrt{\frac{a+x}{a-x}} = \pm 1, \text{ when } x = 0; \ \therefore \text{ the}$$

curve cuts the axis of x at the origin at angles of 45° and 135°. When

$x = -a$, $\frac{dy}{dx} = \infty$; $\therefore$ the curve cuts the axis of x at B at right angles.

$$\frac{d^2y}{dx^2} = \pm\frac{a^2(2a + x)}{(a-x)(a^2-x^2)^{\frac{1}{2}}},$$ which is positive when y is positive,

and negative when y is negative; $\therefore$ one branch has its concavity upward and the other downward.

Ex (4.) The equation to a curve is $xy^2 + 2a^2y - x^3 = 0$, it is required to describe it.

$$y = -\frac{a^2}{x} \pm \left(x^2 + \frac{a^4}{x^2}\right)^{\frac{1}{2}}, \qquad (1)$$

$$\text{or } y\left(y + \frac{2a^2}{x}\right) - x^2 = 0. \qquad (2)$$

By expanding (1) in ascending powers of x, and taking the upper sign, we have

$$y = \frac{1}{2}\frac{x^3}{a^2} - \frac{1}{2^2 . 1 . 2}\frac{x^7}{a^6} + \&c. \qquad (3)$$

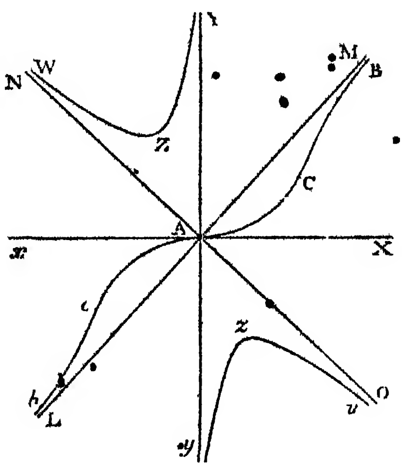
Taking the lower sign, we have

$$y = -\frac{2a^2}{x} - \left(\tfrac{1}{2}\frac{x^3}{a^2} - \frac{1}{2^2 \cdot 1 \cdot 2}\frac{x^7}{a^6} + \&c.\right) \quad (4)$$

Expanding (1) in descending powers of x, and taking the lower sign,

$$y = -\frac{a^2}{x} - x\left(1 + \tfrac{1}{2}\frac{a^2}{x^4} - \frac{1}{2^2 \cdot 1 \cdot 2}\frac{a^4}{x^8} + \&c.\right) \quad (5)$$

In (2) let $x = 0$, then $y = 0$, and $y = -\infty$.



In (3) let x be increased positively, then if x be small, the first term of the series is greater than the sum of all those that follow it, $\therefore$ y is positive, and when $x = \infty$, $y = \infty$; $\therefore$ this branch of the curve lies entirely in the first quadrant, and extends to infinity. It is represented by A C B.

In (5) let $x = \infty$, then $y = -\infty$; and in (4) let $x = 0$, then $y = -\infty$; $\therefore$ this branch of the curve lies entirely in the fourth quadrant, and extends to infinity. It is represented by $y\,z\,w$.

It is obvious that the negative values will be obtained by substituting $-x$ and $-y$ for $+x$ and $+y$ in equations (2), (3), (4), and (5), and as these equations will remain unchanged, the opposite quadrants

are symmetrical; $\therefore$ the investigations of the forms of the curve in the first and fourth are sufficient.

When $x = 0$, $y = \pm - \infty$; $\therefore$ the axis of y is an asymptote to the curve in the fourth quadrant. Also, since

$$y = -\frac{a^2}{x} \pm x\left(1 + \frac{1}{2}\frac{a^2}{x^4} - \frac{1}{2^2 . 1 . 2}\frac{a^4}{x^8} + \&c.\right), \text{ when } x = \infty,$$

$y = \pm x$, which is therefore the equation to the asymptotes LM and NO.

Again, since $y = \dfrac{1}{2}\dfrac{x^3}{a^3} - \dfrac{1}{2^2 . 1 . 2}\dfrac{x^7}{a^6} + \&c.$, $\dfrac{dy}{dx} = \dfrac{3}{2}\dfrac{x^2}{a^3} -$

$\dfrac{7}{2^2 . 1 . 2}\dfrac{x^6}{a^6} + \&c. = 0$, when $x = 0$; $\therefore$ the axis of x is a tangent

to the curves BCA and bcA at the origin A.

Also the minimum values of y in the second and fourth quadrants are $+ a\sqrt{3}$ and $- a\sqrt{3}$, which correspond to the points Z and z; and by the usual process it appears that there is a point of inflexion of the branch BCAb at the origin.

(140.) To trace a curve referred to polar co-ordinates from its equation.

(1.) Let the equation to the curve be reduced if possible to the form $r = f(\theta)$; and take a point for the pole, and a line drawn through it for the axis from which θ is to be measured

(2.) Substitute $\pm n\pi$ for θ, which will determine the points in which the curve cuts the axis.

(3.) Substitute $\pm \frac{1}{2}(2n + 1)\pi$ for θ, which will determine the points in which the radius vector is at right angles to the axis.

(4.) Find the values of θ which render r a maximum or minimum from the equation $\dfrac{dr}{d\theta} = 0$

(5.) Find whether the curve admits of rectilineal or circular asymptotes.

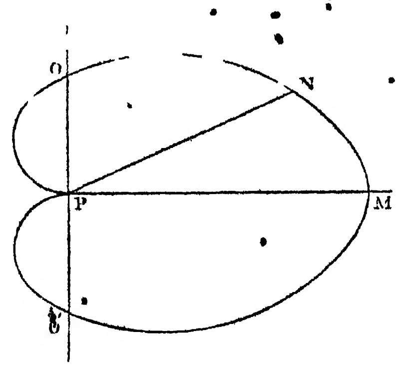
(6.) Determine the singular points of the curve by the rules already given.

Example (1.) The equation to the cardioid is $r = a(1 + \cos. \theta)$: it is required to trace it.

(1.) Let $\pm n\pi = \theta$.

$n = 0$, then $\theta = 0$, and $r = 2a = \text{PM}$;

$n = 1$, then $\theta = \pm \pi \ldots r = 0$.



(2.) Let $\pm \frac{1}{2}(2n + 1)\pi = \theta$.

$n = 0$, then $\theta = \pm \frac{\pi}{2}$. and $r = a$;

$n = 1, \ldots \theta = \pm \frac{3\pi}{2}, \ldots r = a$.

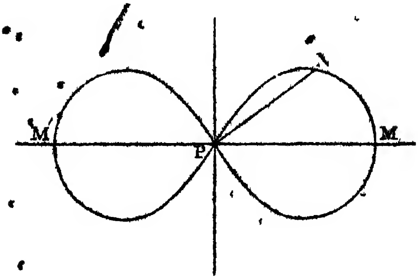
The curve is therefore of the form in the diagram, the point P being a cusp, and PO or PO' $= a$.

Ex. (2.) The equation to the lemniscata of Bernoulli is $r^2 = a^2 \cos. 2\theta$: it is required to describe it.

(1.) Let $\pm n\pi = \theta$.

$n = 0$, then $\theta = 0$, and $r = \pm a = $ PM or PM′;

$n = 1$, then $\theta = \pm \pi$, and $r = \pm a$.



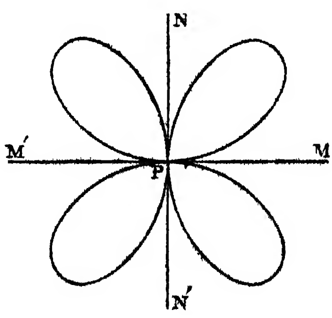
(2.) Let $\pm \frac{1}{2}(2n + 1)\pi = \theta$.

$n = 0$, then $\theta = \pm \frac{\pi}{2}$, and $r = \pm a\sqrt{-1}$;

$n = 1, \ldots \theta = \pm \frac{3\pi}{2}, \ldots r = 0$.

Also, since $2\theta = \cos.^{-1} \frac{r^2}{a^2}$, when $r = 0$, $2\theta = \cos.^{-1} 0$; $\therefore \theta = 45°$ or $135°$. Consequently the curve cuts the axis M′M at P at these angles, and it is obvious that there is a double point at the pole.

Ex. (3.) The equation to a curve is $r = a \sin. 2\theta$: it is required to trace it.



(1.) Let $\pm n\pi = 2\theta$, then $r = 0$ when $\theta = 0$, $\theta = \pm \frac{\pi}{2}$, $\theta = \pm \pi$, and $\theta = \pm \frac{3\pi}{2}$.

When $\theta = \pm 2\pi$ and upward, the same values recur.

(2.) Let $\pm \frac{1}{2}(2n+1)\pi = 2\theta$, then $r = \pm a$ when $\theta = \pm \frac{\pi}{4}$, $\theta = \pm \frac{3\pi}{4}$, $\theta = \pm \frac{5\pi}{4}$, $\theta = \pm \frac{7\pi}{4}$. When $\theta = \pm \frac{9\pi}{4}$ and upward, the same values recur.

It is obvious, therefore, that the curve has four loops as in the figure, and that there is a quadruple point at the pole.

Examples for Practice.

(1.) Prove that the curve whose equation is $y^3 = x^5$ has a point of inflexion at the origin.

(2.) The equation of a curve is $a^2y = 3bx^2 - x^3$: prove that the point $x = b$, $y = \frac{2b^3}{a^2}$, is a point of inflexion.

(3.) The equation of a curve is $y^3 = x^3 - a^3$: prove that the points $x = 0$, $y = -a$, and $x = a$, $y = 0$, are points of inflexion.

(4.) The equation of a curve is $(y - x)^2 = x^9$: prove that the origin is a cusp or point of reflexion.

(5.) The equation of a curve is $(y - b)^2 = (x - a)^3$: prove that the point $x = a$, $y = b$ is a cusp.

(6.) The equation of a curve is $(2x + y - x^2)^2 = (x - 1)^5$: prove that the point $x = 1$, $y = -1$, is a cusp.

(7.) The equation of a curve is $(a^2 - x^2)y^2 = (a^2 + x^2)x^2$: prove that there is a double point at the origin, and that the branches cut the axis of x at angles of 45° and 135°.

(8.) The equation to a curve is $ay^2 = bx^2 + x^3$; prove that there is a double point at the origin, and that the branches cut the axis of x at angles whose tangents are $\left(\frac{b}{a}\right)^{\frac{1}{2}}$ and $-\left(\frac{b}{a}\right)^{\frac{1}{2}}$.

(9.) The equation to a curve is $y^2 = \left(\frac{x-b}{a-x}\right)x^2$: prove that the origin is a conjugate point.

(10.) The equation to a curve is $y^2 = (x+1)(x-1)^{\frac{1}{2}} + 1$: prove that the points $x = -1$, $y = \pm 1$, are conjugate points.

(11.) The equation to a curve is $y - b = (x - a)\left(\dfrac{x}{a}\right)^{\frac{1}{2}}$: prove that the curvature is a maximum at the point $x = 0$, $y = b$.

(12.) The equation of a curve is $x^4 + a^3y + b^3x = 0$: it is required to trace it.

(13.) The equation of a curve is $y^2 - 2xy + 3x^2 - 10x + 12 = 0$: it is required to trace it.

(14.) The equation of a curve is $xy = a(x + y)$: it is required to trace it.

(15.) Trace the curve whose equation is $r = a \cos. \theta$.

(16.) Trace the curve whose equation is $r = a \tan. \theta$.

(17.) Trace the curve whose equation is $r = 2a \dfrac{\sin.^2 \theta}{\cos. \theta}$.

(18.) Trace the curve whose equation is $r = a \dfrac{\sin.^3 \theta}{\cos. \theta}$.

CHAPTER XV.

CURVE SURFACES AND CURVES OF DOUBLE CURVATURE.

TANGENCIES AND DIFFERENTIATION OF VOLUMES AND SURFACES.

(141.) When a curve surface is referred to three rectangular axes of co-ordinates, its equation is in general of one of the forms $z = f(x, y)$ or $f(x, y, z) = 0$, where x, y, and z are the co-ordinates of any point in it.

(142.) To find the equation to a plane touching a curve surface at any point.

Let $z = f(x, y)$ be the equation to the curve surface, and $z' = Ax' + By' + C$ that of the tangent plane. Then since at the point of contact $z = Ax + By + C$, we have $z' - z = A(x' - x) + B(y' - y)$.

(1.) Let a plane pass through the point of contact parallel to the plane xz, then we have $y = y'$ for the intersections of this plane with the tangent plane and curve surface; $\therefore z' - z = A(x' - x.)$ But since the section of the tangent plane must be a tangent to the section of the curve surface, we have $\frac{dz}{dx} = A$; $\therefore z' - z = \frac{dz}{dx}(x' - x)$.

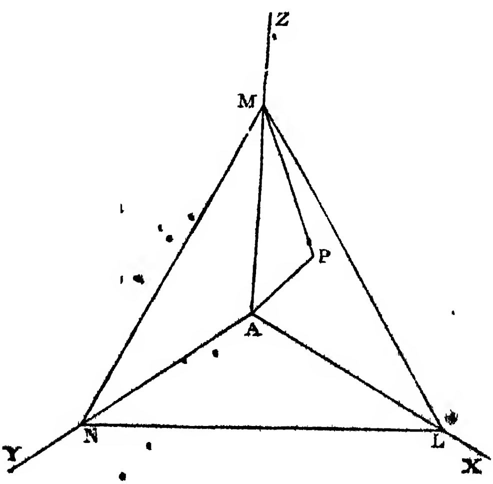
(2) Let a plane pass through the point of contact parallel to the plane yz, then $x = x'$ for the intersections of this plane with the tangent plane and curve surface. $\therefore z' - z = B(y' - y)$, and $\frac{dz}{dy} = B$, as in (1); $\therefore z' - z = \frac{dz}{dy}(y' - y)$. Hence $z' - z = \frac{dz}{dx}(x' - x) +$

$\frac{dz}{dy}(y' - y)$, or $z' - z = p(x' - x) + q(y' - y)$ is the equation required, p and q being the partial differential coefficients of z, obtained from the equation to the surface by supposing y and x respectively constant.

(143.) To find the angles which a plane touching a curve surface at any point makes with the co-ordinate planes.

Let α, β, and γ be the angles which the tangent plane makes with the planes $y\,z$, $x\,z$, and $x\,y$ respectively; then since x, y, and z are the co-ordinates of the point of contact, we have $x \cos.\alpha + y \cos.\beta + z \cos.\gamma = a$. (Vide *Waud's Algebraical Geometry*, page 211.) $\therefore p = \frac{dz}{dx} = -\frac{\cos.\alpha}{\cos.\gamma}$, $q = \frac{dz}{dy} = -\frac{\cos.\beta}{\cos.\gamma}$; $\therefore 1 + p^2 + q^2 = 1 + \frac{\cos.^2\alpha}{\cos.^2\gamma} + \frac{\cos.^2\beta}{\cos.^2\gamma} = \frac{1}{\cos.^2\gamma}$; $\therefore \cos.\gamma = \frac{1}{\sqrt{1 + p^2 + q^2}}$, $\cos.\alpha = -\frac{p}{\sqrt{1 + p^2 + q^2}}$, and $\cos.\beta = -\frac{q}{\sqrt{1 + p^2 + q^2}}$.

(144.) To find the length of the perpendicular from the origin on the tangent plane.



Let A be the origin A X, A Y, and A Z be the rectangular co-ordinates, and L M N the tangent plane.

Draw A P perpendicular to the plane L M N and join M P.

It can easily be demonstrated by elementary geometry, that the angle M A P = the angle which the plane M N L makes with the plane X Y, which is equal to γ. But A P = A M cos. M A P, and since the equation of the tangent plane is $z' - z = p(x' - x) + q(y' - y)$, when $x' = 0$ and $y' = 0$, we have $z' = \text{A M} = z - px - qy$; $\therefore$ A P = P =

$$\frac{z - px - qy}{\sqrt{1 + p^2 + q^2}}.$$

(145.) To find the equation of the normal at any point in a curve surface.

Let $x' = \alpha z' + a$, and $y' = \beta z' + b$, be the equations of the projections of the normal line on the planes of xz and yz respectively; then since the normal passes through the point x, y, z, these equations become $x = \alpha z + a$, and $y = \beta z + b$; $\therefore x' - x = \alpha (z' - z)$, and $y' - y = \beta (z' - z)$. But the equation to the tangent plane is $z' - z =$

$\frac{dz}{dx}(x' - x) + \frac{dz}{dy}(y' - y)$, and since the normal and tangent are at

right angles to each other, we have $\alpha + \frac{dz}{dx} = 0$, and $\beta + \frac{dz}{dy} = 0$.

Substituting these values of α and β in the equations already found,

we have $x' - x + \frac{dz}{dx}(z' - z) = 0$, and $y' - y + \frac{dz}{dy}(z' - z) = 0$,

which together determine the normal.

(146.) To find the length of that portion of the normal intercepted between the surface and any of the co-ordinate planes.

Let x', y', z' be the co-ordinates of any point in the normal, and x, y, z those of the point where it meets the surface, then

$d = \sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2}$—(*Waud's Algebraical Geometry*, page 202) $= (z'-z)\sqrt{1 + \left(\frac{x'-x}{z'-z}\right)^2 + \left(\frac{y'-y}{z'-z}\right)^2}$. But since $x' - x + \frac{dz}{dx}(z'-z) = 0$ (145), $\left(\frac{x'-x}{z'-z}\right)^2 = \left(\frac{dz}{dx}\right)^2$. For a similar reason $\left(\frac{y'-y}{z'-z}\right)^2 = \left(\frac{dz}{dy}\right)^2$; $\therefore\ d = (z'-z)\sqrt{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2}$.

Now at the point where the normal meets the plane of xy, $z' = 0$; $\therefore\ d = -z\sqrt{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2} = z\sqrt{1 + p^2 + q^2}$ if the sign be neglected.

In a similar manner it may be demonstrated that if d' and d'' represent the portions of the normal intercepted between the point in the surface, and the planes yz and xz, we have $d' = \frac{x}{p}\sqrt{1 + p^2 + q^2}$, and $d'' = \frac{y}{q}\sqrt{1 + p^2 + q^2}$.

EXAMPLE (1.) The equation to a curve surface is $xyz = m^3$: find the equation to its tangent plane, the intercepts on the co-ordinate axes, and the volume of the pyramid included between the tangent plane and the co-ordinate planes.

$$z = \frac{m^3}{xy};\quad \therefore\ \frac{dz}{dx} = -\frac{m^3}{x^2y} = -\frac{z}{x},\quad \frac{dz}{dy} = -\frac{z}{y};$$

$$\therefore\ (z'-z)\,xy + (x'-x)\,yz + (y'-y)\,xz = 0,$$

$$x'yz + y'xz + z'xy = 3\,xyz;$$

$$\therefore\ \frac{x'}{x} + \frac{y'}{y} + \frac{z'}{z} = 3, \text{ which is the equation to the tangent}$$

plane.

Again, the intercepts on the co-ordinate axes $x'_0 = 3x$, $y'_0 = 3y$ and $z'_0 = 3z$.

Also the volume of the pyramid included between the tangent plane and the co-ordinate planes is equal to $\frac{9xyz}{2} = \frac{9m^3}{2}$.

Ex. (2.) The equation of an ellipsoid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$: find the equation of its tangent plane, the intercepts on the axes, and the length of the perpendicular from the origin on the plane.

$p = -\frac{c^2x}{a^2z}$, $q = -\frac{c^2y}{b^2z}$. Substituting these in the equation,

$z' - z = p(x' - x) + q(y' - y)$, and $z' - z + \frac{c^2x}{a^2z}(x' - x) + \frac{c^2y}{b^2z}(y' - y) = 0$.

$\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, which is the equation of the tangent.

Let y' and $z' = 0$, then $x' = \frac{a^2}{x}$. In a similar manner it appears that

$$y' = \frac{b^2}{y} \text{ and } z' = \frac{c^2}{z}.$$

Also, since $P = \frac{z - px - qy}{\sqrt{1 + p^2 + q^2}} = \frac{1}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}$; $\therefore \frac{1}{P^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}$.

Ex. (3.) The equation to the *heliçoïde gauche* is $x \cos.\left(\frac{2\pi z}{h}\right)$ —

$y \sin.\left(\frac{2\pi z}{h}\right) = 0$, find the equation to its tangent plane, and the perpendicular from the origin on that plane.

$$x \cos.\left(\frac{2\pi z}{h}\right) - y \sin.\left(\frac{2\pi z}{h}\right) = 0; \;\therefore \sin.\left(\frac{2\pi z}{h}\right) = \frac{x}{\sqrt{x^2+y^2}};$$

$$\therefore z = \frac{h}{2\pi}\sin.^{-1}\frac{x}{\sqrt{x^2+y^2}},\; p = \frac{h}{2\pi}\times\frac{y}{x^2+y^2},\text{ and } q = -\frac{h}{2\pi}\frac{x}{x^2+y^2},$$

$$\therefore z' - z = p(x'-x) + q(y'-y) = \frac{h}{2\pi}\frac{y}{x^2+y^2}(x'-x) - \frac{h}{2\pi}\frac{x}{x^2+y^2}$$

$(y'-y)$; $\therefore h(xy'-yx') + 2\pi(x^2+y^2)z' = 2\pi(x^2+y^2)z$, which is the equation to the tangent plane.

Also $P = \frac{z-px-qy}{\sqrt{1+p^2+q^2}} = \frac{2\pi(x^2+y^2)^{\frac{1}{2}}z}{(h^2+4\pi^2(x^2+y^2))^{\frac{1}{2}}} = \frac{2\pi rz}{(h^2+4\pi^2r^2)^{\frac{1}{2}}}$

when $r = \sqrt{x^2+y^2}$.

Ex. (4.) It is required to draw a normal to an ellipsoid, and to find the lengths of the portions of it intercepted between the surface of the ellipsoid and the co-ordinate planes.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1.$$

$$\therefore \frac{dz}{dx} = -\frac{c^2x}{a^2z},\quad \frac{dz}{dy} = -\frac{c^2y}{b^2z};$$

$$\therefore x' - x = \frac{c^2x}{a^2z}(z'-z),$$

$$y' - y = \frac{c^2y}{b^2z}(z'-z).$$

Again $d = z\sqrt{1+p^2+q^2}$ (164) $= c^2\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}} = \frac{c^2}{P}$, (Ex. 2.)

In a similar manner it appears that

$$d' = \frac{a^2}{P}, \text{ and } d'' = \frac{b^2}{P}.$$

(147.) When a generating point not only continually changes its direction, but also the plane in which it moves, it describes a curve of double curvature.

(148.) To draw a tangent line to a curve of double curvature.

Let $y = f(x)$ and $z = \varphi(x)$ be the equations of the projections of the curve on the planes of xy and xz respectively, and x', y', z' the co-ordinates of any point in the tangent line, then the co-ordinates of the projections of this line, on the planes of xy and xz, will be x', y' and x', z'; and since the projections of the tangent are tangents to the projections of the curve, we have $y' - y = \frac{dy}{dx}(x' - x)$, and $z' - z = \frac{dz}{dx}(x' - x)$ which are the equations to the tangent line at any point of a curve of double curvature.

(149.) To find the equation of the normal plane at any point of a curve of double curvature.

Let $z' = \text{A}x' + \text{B}y' + \text{C}$ be the equation of the normal, then, since it passes through the point x, y, z, we have

$$z = \text{A}x + \text{B}y + \text{C}, \text{ and } \therefore z' - z = \text{A}(x' - x) + \text{B}(y' - y);$$

and when z' and y' are made successively $= 0$, we have $y' - y = -\frac{\text{A}}{\text{B}}(x' - x) - \frac{z}{\text{B}}$, and $z' - z = \text{A}(x' - x) - \text{B}y$, for the equations of the traces of the normal plane on the planes of xy and xz respectively. But these traces must be perpendicular to the lines, whose equations are $y' - y = \frac{dy}{dx}(x' - x)$, and $z' - z = \frac{dz}{dx}(x' - x)$; $\therefore \frac{\text{A}}{\text{B}} = \frac{dx}{dy}$, and

$$\text{A} = -\frac{dx}{dz}, \therefore \text{B} = -\frac{dy}{dz}.$$

Hence $x' - x + (y' - y)\dfrac{dy}{dx} + (z' - z)\dfrac{dz}{dx} = 0$ is the equation to the normal at any point in a curve of double curvature.

EXAMPLE (1.) The equations to the helix are $x = a\cos.\dfrac{z}{h}$ and $y = a\sin.\dfrac{z}{h}$: find the equations to its tangent line and normal plane at the point x, y, z.

$$\frac{dx}{dz} = -\frac{a}{h}\sin.\frac{z}{h} = -\frac{y}{h};\ \therefore\ x' - x = -\frac{y}{h}(z' - z) \;\&\; h(x' - x) + y(z' - z) = 0,$$

$$\frac{dy}{dz} = \frac{a}{h}\cos.\frac{z}{h} = \frac{x}{h};\ \therefore\ y' - y = \frac{x}{h}(z' - z) \;\&\; h(y' - y) - x(z' - z) = 0,$$

which are the equations to the tangent, and $xy' - x'y + h(z' - z) = 0$, is the equation to the normal.

Ex. (2.) A curve is formed by the intersection of a sphere and ellipsoid, it is required to find the equations to its tangent and normal.

The equation of the ellipsoid is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, and of the sphere $x^2 + y^2 + z^2 = r^2$; $\therefore\ y^2 = b^2 - \dfrac{b^2x^2}{a^2} - \dfrac{b^2z^2}{c^2}$, and $y^2 =$ $r^2 - x^2 - z^2$; $\therefore\ \dfrac{a^2 - b^2}{a^2}x^2 = r^2 - b^2 + \dfrac{b^2 - c^2}{c^2}z^2$, and $\dfrac{dx}{dz} = \dfrac{a^2}{c^2}\,\dfrac{b^2 - c^2}{a^2 - b^2}\cdot\dfrac{z}{x}$.

In a similar manner it appears that $\dfrac{dy}{dz} = \dfrac{b^2}{c^2}\,\dfrac{c^2 - a^2}{a^2 - b^2}\cdot\dfrac{z}{y}$. By substituting these in the equations to the tangent, we obtain

$$\frac{x}{a^2}\cdot\frac{x' - x}{b^2 - c^2} = \frac{y}{b^2}\cdot\frac{y' - y}{c^2 - a^2} = \frac{z}{c^2}\cdot\frac{z' - z}{a^2 - b^2}.$$

Again, the equation to the normal is

$$a^2(b^2-c^2)\frac{x'-x}{x}+b^2(c^2-a^2)\frac{y'-y}{y}+c^2(a^2-b^2)\frac{z'-z}{z}=0.$$

(150.) To find the differentials of the volume and surface of a solid bounded by co-ordinate planes, and a curve surface whose equation is given.

Let $z = f(x, y)$ be the equation of the surface; then $v = \varphi(x, y)$ will be that of the volume of the solid D F R N M E P.

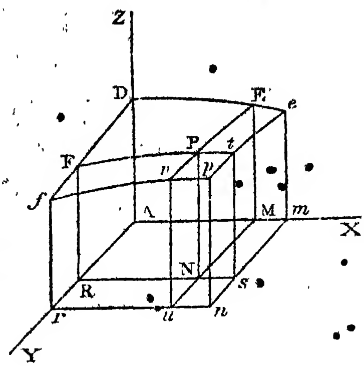
Let x and y become equal to $x + h$ and $y + k$ respectively, then the increment on the solid is contained by the parallel planes E R $s\,t$ and $f\,r\,n\,p$, and the parallel planes P N M E and $t\,s\,m\,e$ =

$$\frac{dv}{dx}h+\frac{dv}{dy}k+\frac{d^2v}{dx^2}\frac{h^2}{1\,.\,2}+\frac{d^2v}{dx\,dy}hk+\frac{d^2v}{dy^2}\frac{k^2}{1\,.\,2}+\&\text{c.} \qquad (1.)$$

Let x alone vary, then the increment on the solid is contained by the parallel planes P N M E and $t\,s\,m\,e$, and

$$=\frac{dv}{dx}h+\frac{d^2v}{dx^2}\frac{h^2}{1\,.\,2}+\&\text{c.} \qquad (2.)$$

But if y alone vary, we have the increment on the solid contained between the parallel planes F R N P and $f\,r\,u\,v$, and



$$=\frac{dv}{dy}k+\frac{d^2v}{dy^2}\frac{k^2}{1\,.\,2}+\&\text{c.} \qquad (3.)$$

By subtracting the sum of (2) and (3) from (1), we have the solid

$$u\,t=\frac{d^2v}{dx\,dy}hk+\&\text{c.}$$

$$\therefore\ \frac{d^2v}{dx\,dy}+\&\text{c.}=\frac{\text{the solid } ut}{hk}.$$

But in a limiting state, the solid ut is a rectangular parallelopiped $= zhk$; $\therefore \frac{d^2v}{dx\,dy} = z$, and $d^2v = z\,dx\,dy$.

If we differentiate also in regard to z, we shall have $d^3v = dx\,dy\,dz$.

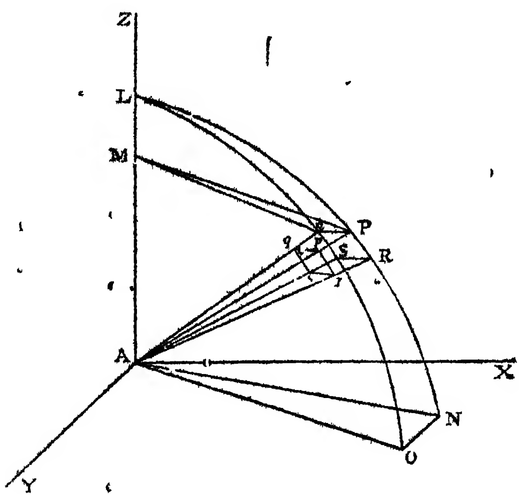
If S = the curve surface, D F P E, we shall have in a similar manner

$$\frac{d^2S}{dx\,dy} + \&c. = \frac{\text{the surface } P\,v\,p\,t}{hk}.$$

But this surface in a limiting state coincides with the tangent plane, and is therefore equal to the base N $u\,n\,s$ $\times$ secant of the inclination of that plane to the plane xy. But $\sec.\gamma = \frac{1}{\cos\gamma} = \sqrt{1+p^2+q^2}$ (143); $\therefore$ the tangent plane $= hk\sqrt{1+p^2+q^2}$; $\therefore \frac{d^2S}{dx\,dy} = \sqrt{1+p^2+q^2}$, and $d^2S = dx\,dy\sqrt{1+p^2+q^2}$.

If we differentiate also in regard to z, we shall have $d^3S = \frac{r+t}{\sqrt{1+p^2+q^2}}\,dx\,dy\,dz$, r and t being equal to $\frac{d^2z}{dx^2}$ and $\frac{d^2z}{dy^2}$ respectively.

(151.) To find the differentials of the volume and surface of a solid referred to polar co-ordinates.



Let Q R and qr be portions of concentric surfaces intercepted by planes passing through the axis of z perpendicular to the plane of $x\,y$, and the planes A P Q and A R S perpendicular to the former through the origin A, the included angles being indefinitely small in each case. Let A P $= r$, L A P $= \theta$, and X A N $= \varphi$. Draw P M perpendicular to A L, and join M Q, then P Q $=$ P M $d\varphi = r \sin.\,\theta\, d\varphi$, P R $= r\,d\theta$ and P $p = dr$. But the solid P s which is represented by $d^3 v =$ PQ $\times$ PR $\times$ Pp in a limiting state $= r^2 \sin.\,\theta\, dr\, d\theta\, d\varphi$.

It also appears that the surface P Q S R $= d^2 S = r^2 \sin.\,\theta\, d\theta\, d\varphi$; $\therefore d^3 S = 2r \sin.\,\theta\, dr\, d\theta\, d\varphi$.

(152.) To find the differential of the arc of a curve of double curvature

Let $y = f(x)$ and $z = \varphi(x)$ be the equations of the projections of the curve on the planes of xy and xz respectively, and when x becomes equal to $x + h$, let $y = y + k$, and $z = z + l$, and let $r =$ the chord of two consecutive points in the curve, then $r^2 = h^2 + k^2 + l^2$. But since $y = f(x)$,

$$k = \frac{dy}{dx} h + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \text{\&c.}\quad \text{For a similar reason}$$

$$l = \frac{dz}{dx} h + \frac{d^2z}{dx^2}\frac{h^2}{1.2} + \text{\&c.}$$

$$\therefore \frac{r^2}{h^2} = 1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2 + \text{M}\,h + \text{N}\,h^2 + \text{\&c.}$$

$$\text{But } \left(\frac{ds}{dx}\right)^2 = \text{the limit of } \frac{r^2}{h^2} = 1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2;$$

$$\therefore \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}.$$

CHAPTER XVI.

OSCULATION AND RADIUS OF CURVATURE.

(153.) To find the conditions necessary to the different orders of contact in osculating curve surfaces.

Let $z = f(x, y)$ and $z' = \varphi(x', y')$, be the equations to the two surfaces referred to the same co-ordinate axes, and let x and x' become equal to $x + h$ and $x' + h$, and y and $y' = y + k$, and $y' + k$ respectively, then the new values of z and z' are

$$z + \frac{dz}{dx} h + \frac{dz}{dy} k + \frac{1}{2}\left(\frac{d^2 z}{dx^2} h^2 + 2 \frac{d^2 z}{dx\, dy} h k + \frac{d^2 z}{dy^2} k^2\right) + \&c.$$

$$\text{and } z' + \frac{dz'}{dx'} h + \frac{dz'}{dy'} k + \frac{1}{2}\left(\frac{d^2 z'}{dx'^2} h^2 + 2 \frac{d^2 z'}{dx'\, dy'} h k + \frac{d^2 z'}{dy'^2} k^2\right) + \&c.$$

$$= z + p h + q k + \tfrac{1}{2} (r h^2 + 2 s h k + t k^2) + \&c.$$

$$\text{and } = z' + P h + Q k + \tfrac{1}{2} (R h^2 + 2 S h k + T k^2) + \&c.$$

Let the surface whose equation is $z' = \varphi(x', y')$ contain a certain number of constants, and let the value of one of them be determined by the condition $z' = z$, and substituted in the original equation, then $x' = x$ and $y' = y$, and the two surfaces will have a common point, x, y, z. Let two other constants be determined by the conditions $P = p$ and $Q = q$, and their values substituted in the equation $z' = \varphi(x', y')$, and the two surfaces will have contact of the *first order*; and if three more constants be determined by the conditions $R = r$, $S = s$, and $T = t$, and their values also substituted in the same equation, the two surfaces will have contact of the *second order*. It hence appears that contact of the first order requires three disposable constants,

and contact of the second order six. $\therefore$ contact of the n^{th} order will require $\frac{(n+1)(n+2)}{2}$ constants.

Example (1.) To find the order of contact which a given plane may have with a given surface.

Let $z' = Ax' + By' + C$ (1) be the equation of the plane, then since it passes through the point x, y, z,

$$z = Ax + By + C. \qquad (2)$$

$$\therefore z' - z = A(x' - x) + B(y' - y). \qquad (3)$$

But $\frac{dz'}{dx'} = A = P$, and $\frac{dz'}{dy'} = B = Q$, from (1),

and $\frac{dz}{dx} = A = p$, and $\frac{dz}{dy} = B = q$, from (2).

$\therefore P = p$ and $Q = q$, and equation (3) becomes

$z' - z = p(x' - x) + q(y' - y)$, which is the equation of a tangent plane already found $\therefore$ a tangent plane has contact of the first order with a given surface.

Ex. (2.) To find the degree of contact which a sphere may have with a given surface.

Let $(x' - \alpha)^2 + (y' - \beta)^2 + (z' - \gamma)^2 = \delta^2$ be the equation of the sphere, then $\frac{dz'}{dx'} = -\frac{x' - \alpha}{z' - \gamma} = P$, and $\frac{dz'}{dy'} = -\frac{y' - \beta}{z' - \gamma} = Q$.

But since the sphere passes through the point x, y, z, we have $z' = z$, $P = p$, and $Q = q$.

$$\therefore (x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = \delta^2,$$

$$x - \alpha + p(z - \gamma) = 0,$$

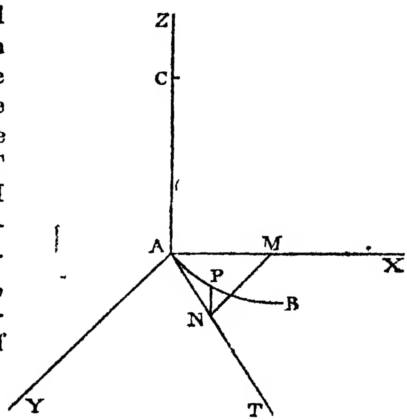
$y - \beta + q(z - \gamma) = 0$, which equations enable us to determine any three of the constants $\alpha, \beta, \gamma, \delta$.

The last two of these equations are those of a normal at the point x, y, z, whose current co-ordinates are α, β, γ. It hence appears that the centre of the sphere is always in the normal passing through the point of contact.

It also appears, that since the equation of the sphere contains four disposable constants, and there are only three equations for determining them, the number of spheres which may have contact of the first order at a given point, in a given surface, is infinite.

(154.) To find the radius of curvature of any section of a curve surface made by a plane passing through the normal at any point.

Let A, the origin of co-ordinates, be the point, and let the normal coincide with the axis of z. Let any plane pass through A Z, and let its intersection with the given surface be A B, and, with the plane of xy, A T. Then, since X A Y is a tangent plane to the curve surface at A, the line A T is a tangent to A B. Let A M $= h$, and M N $= k$, be the co-ordinates of N. Draw N P parallel to A Z, and let N P $= l$, and A C $= \varrho$ the radius of curvature, then $\varrho = \frac{1}{2}$ the limit of $\frac{\text{AN}^2}{\text{NP}} = \frac{1}{2}$ the limit of $\frac{h^2 + k^2}{l}$.



But $l = ph + qk + \frac{1}{2}(rh^2 + 2shk + tk^2) + \&c.$; and since the plane of xy is a tangent at A, $p = 0$ and $q = 0$; $\therefore \varrho = \frac{1}{2}$ the

$$\text{limit of } \frac{2(h^2+k^2)}{rh^2 + 2shk + tk^2 + \&c.} = \frac{1}{2} \text{ the limit of } \frac{2\left(1 + \left(\frac{k}{h}\right)^2\right)}{r + 2s\left(\frac{k}{h}\right) + \left(\frac{k}{h}\right)^2 + \&c.}$$

$$= \frac{1 + \tan.^2 \theta}{r + 2 s \tan. \theta + t \tan.^2 \theta} = \frac{1}{r \cos.^2 \theta + 2 s \cos. \theta \sin. \theta + t \sin^2 \theta},$$

where $\theta =$ the angle which the normal section makes with the plane of $x z$.

Cor. Let $\varrho' =$ the radius of curvature of a section inclined to the plane of $x z$ at an angle of $90° + \theta$, then

$$\varrho' = \frac{1}{r \sin.^2 \theta - 2 s \sin. \theta \cos. \theta + t \cos.^2 \theta}; \therefore \frac{1}{\varrho} + \frac{1}{\varrho'} = r + t =$$

a constant quantity.

(155.) To find the normal sections of greatest and least curvature at any point of a curve surface.

$$\varrho = \frac{1}{r \cos.^2 \theta + 2 s \cos. \theta \sin. \theta + t \sin^2 \theta}.$$

Let $u = \frac{1}{\varrho}$, then $\frac{du}{d\theta} = - 2 r \sin. \theta \cos. \theta - 2 s \sin.^2 \theta + 2 s \cos.^2 \theta$

$+ 2 t \sin. \theta \cos. \theta = 0,$

$$\therefore (t - r) \tan. \theta + s (1 - \tan^2 \theta) = 0,$$

$$\text{and } \tan. \theta = \frac{t - r \pm \sqrt{(t - r)^2 + 4 s^2}}{2 s}. \quad (1)$$

The upper sign gives the value of θ corresponding to the greatest curvature, and the lower to the least; then, if θ_1 and θ_2 represent these values respectively, we shall have from (1) $\tan. \theta_1 \tan. \theta_2 = - 1$.

But $\tan. \theta_1 \tan. (90 + \theta_1) = - 1$; $\therefore$ the sections of greatest and least curvature are at right angles to each other.

Again, we have $\varrho = \frac{1 + \tan.^2 \theta}{r + 2 s \tan. \theta + t \tan.^2 \theta}$ by (154) $= \frac{1 + \cot.^2 \theta}{r \cot.^2 \theta + 2 s \cot. \theta + t}$;

$\therefore \frac{1}{\varrho} = \frac{1}{1 + \cot.^2 \theta} (r \cot.^2 \theta + 2 s \cot. \theta + t)$. But since $(t - r)$

$\tan.\,\theta + s\,(1 - \tan.^2\theta) = 0,\ t = s\tan.\,\theta - s\cot.\,\theta + r;\ \therefore \dfrac{1}{\rho} =$

$\dfrac{1}{1 + \cot.^2\theta}\left(r\,(1 + \cot.^2\theta) + s\,(\tan.\,\theta + \cot.\,\theta)\right) = r + s\tan.\,\theta =$

$r + \dfrac{t - r \pm \sqrt{(t-r)^2 + 4s^2}}{2} = \dfrac{t + r \pm \sqrt{(t-r)^2 + 4s^2}}{2}$. Hence, if

ρ_1 and ρ_2 represent the radii of greatest and least curvature respectively, we shall have

$$\rho_1 = \frac{2}{t + r - \sqrt{(t-r)^2 + 4s^2}},\ \text{and}\ \rho_2 = \frac{2}{t + r + \sqrt{(t-r)^2 + 4s^2}}.$$

(156.) To express the radius of curvature of any normal section of a curve surface in terms of the radii of curvature of the normal sections of greatest and least curvature.

Let the axes of x and y be taken in the planes of greatest and least curvature, then, since $(t - r)\tan.\,\theta + s\,(1 - \tan.^2\theta) = 0$ (155),

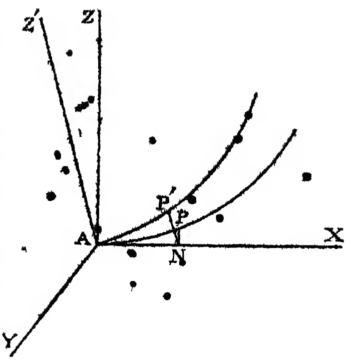
$s = 0;\ \therefore \rho_1 = \dfrac{1}{r}$, and $\rho_2 = \dfrac{1}{t}$; and if a normal section make an angle φ with the plane of $x\,z$, we shall have

$$\rho = \frac{1 + \tan.^2\varphi}{r + 2s\tan.\,\varphi + t\tan.^2\varphi} = \frac{\sec.^2\varphi}{t\tan.^2\varphi + r} = \frac{1}{t\sin.^2\varphi + r\cos.^2\varphi}$$

$$= \frac{1}{\frac{1}{\rho_2}\sin.^2\varphi + \frac{1}{\rho_1}\cos.^2\varphi} = \frac{\rho_1\rho_2}{\rho_1\sin.^2\varphi + \rho_2\cos.^2\varphi}.$$

(157.) To find the radius of curvature at any point in an oblique section of a curve surface.

Let A P represent an oblique section through A, and let A P′ be a normal section through the same point. Take the axis of x a tangent to A P at A, it shall also be a tangent to A P′. Let A Z and A Z′ be perpendicular to A X in the planes of A P and A P′ respectively, and let Z A Z′ $= \theta$, A N $= h$, and ϱ and ϱ' the radii of curvature of A P and A P′ at A; then $\frac{\varrho}{\varrho'}$ = limit of



$$\frac{NP'}{NP} = \text{limit of } \frac{z' \sec.\theta}{z} = \sec.\theta \times \text{limit of } \frac{\frac{r h^2}{1.2} + s h k + \frac{t k^2}{1.2} + \ldots}{\frac{r h^2}{1.2} + \frac{dr}{dx}\frac{h^3}{1.2.3} + \ldots}$$

$$= \sec.\theta \times \text{limit of } \frac{r + 2s\left(\frac{k}{h}\right) + t\left(\frac{k}{h}\right)^2 + \ldots}{r + \frac{dr}{dx}\frac{h}{3} + \ldots} = \sec.\theta$$ (as the limit of $\frac{k}{h} = 0$, since A X is a tangent to the projection of A P′ on the plane of X Y); $\therefore\ \varrho' = \varrho \cos.\theta$.

Hence ϱ' is the projection of ϱ on the plane of the oblique section, which property is the theorem of Meusnier.

Example (1.) To find the radius of curvature of any normal section at the extremity of the axis of z in an ellipsoid.

Let the point be taken as the origin, then the plane of xy is a tangent to the curve surface, since the axis of z coincides with the normal; $\therefore\ z = 0$, $p = 0$, and $q = 0$ at the given point.

But $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{(z-c)^2}{c^2} = 1$. Hence $\frac{1}{a^2} + \frac{p^2}{c^2} + \frac{z-c}{c^2} r = 0$;

$\therefore\ r = \frac{c}{a^2}$.

In a similar manner it appears that $t = \frac{c}{b^2}$ and $s = 0$;

$$\therefore \varrho_{,} = \frac{1}{r} = \frac{a^2}{c} \text{ and } \varrho_2 = \frac{b^2}{c}. \quad \text{But } \varrho = \frac{\varrho_{,}\,\varrho_2}{\varrho_{,} \sin.^2 \varphi + \varrho_2 \cos.^2 \varphi}.$$

$$\therefore \varrho = \frac{a^2 b^2}{c\,(a^2 \sin.^2 \varphi + b^2 \cos.^2 \varphi)} = \frac{a^2}{c} \text{ when } a = b, \text{ which is the}$$

case when the ellipsoid becomes a spheroid of revolution about the axis of z.

Ex. (2.) To find the radius of curvature of a normal section through any point in an oblate spheroid.

Let λ be the angle which the normal at the given point makes with the major axis, then

$$\varrho_{,} = \frac{a\,(1 - e^2)}{(1 - e^2 \sin.^2 \lambda)^{\frac{3}{2}}} \text{ and } \varrho_2 = \frac{a}{(1 - e^2 \sin.^2 \lambda)^{\frac{1}{2}}}.$$

Let φ be the angle which the section makes with the normal plane whose radius at the given point is $\varrho_{,}$, then

$$\varrho = \frac{\varrho_{,}\,\varrho_2}{\varrho_{,} \sin.^2 \varphi + \varrho_2 \cos.^2 \varphi} = \frac{a}{\sqrt{1 - e^2 \sin.^2 \lambda}} \cdot \frac{1 - e^2}{1 + e^2 \cos.^2 \lambda \cos.^2 \varphi - e^2}.$$

(158.) If a line be traced on a curve surface such that the normal to the surface at every point of it is intersected by the consecutive normal, it is called a line of curvature.

(159.) To determine the lines of curvature through any point in a given surface.

Let $x\,y\,z$ be a point in the surface referred to rectangular co-ordinates, then the equations to the normal at that point are $x' - x +$

$$p\,(z' - z) = \text{M} = 0, \text{ and } y' - y + q\,(z' - z) = \text{N} = 0, \qquad (1)$$

Let x and y become equal to $x + h$ and $y + k$ respectively; then the equations to the normal at the corresponding point will be

$$M + \frac{dM}{dx}h + \frac{dM}{dy}k + \&c. = 0, \text{ and } N + \frac{dN}{dx}h + \frac{dN}{dy}k + \&c. = 0.$$

But $M = 0$ and $N = 0$; $\therefore$ these equations become

$$\frac{dM}{dx} + \frac{dM}{dy}\frac{k}{h} + \&c. = 0, \text{ and } \frac{dN}{dx} + \frac{dN}{dy}\frac{k}{h} + \&c. = 0.$$

But as the normals intersect and are consecutive, k and h are dependent, and $h = 0$.

$$\therefore \frac{dM}{dx} + \frac{dM}{dy}\frac{dy}{dx} = 0, \text{ and } \frac{dN}{dx} + \frac{dN}{dy}\frac{dy}{dx} = 0. \quad . \quad . \quad (2)$$

Again, x', y, z', the co-ordinates of the point of intersection of the normals, must have the same values in all the equations. If, therefore, we substitute the values of M and N in (2), as found from (1), we shall have

$$1 + p\left(p + q\frac{dy}{dx}\right) + (z - z')\left(r + s\frac{dy}{d.}\right) = 0; \qquad (3)$$

$$\frac{dy}{dx} + q\left(p + q\frac{dy}{dx}\right) + (z - z')\left(s + t\frac{dy}{dx}\right) = 0; \qquad (4)$$

and by eliminating $z - z$ we have

$$((1 + q^2)s - pqt)\left(\frac{dy}{dx}\right)^2 + ((1+q^2)r - (1+p^2)t)\frac{dy}{dx} - ((1+p^2)s - pqr)$$

$= 0$, which is the differential equation of the projection of the lines of curvature. But $\frac{dy}{dx}$ is the tangent of the angle which the line joining two consecutive points makes with the axis of x; and as it is of two dimensions, it is obvious that there are two lines of curvature through any point in the given surface.

(160.) To find the radii of curvature at any point of a curve surface, in terms of the co-ordinates of that point.

Let x, y, z be the co-ordinates of the given point, and x', y', z', those

of either of the points in the normal corresponding to the centres of curvature, then

$\varrho^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2$. But $x' - x = -(z' - z)\,p$ and $y' - y = -(z' - z)\,q$ (145); $\therefore \varrho^2 = (z' - z)^2\,(1 + p^2 + q^2)$, and $z' - z =$

$\dfrac{\varrho}{\sqrt{1 + p^2 + q^2}} = \dfrac{\varrho}{k}$. But by eliminating $\dfrac{dy}{dx}$ in (3) and (4) of (159), we have

$$(z' - z)^2\,(rt - s^2) - (z' - z)\,\big((1 + q^2)\,r - 2pqs + (1 + p^2)\,t\big) + k^2 = 0\,;$$

$$\therefore \varrho^2\,(rt - s^2) - \varrho k\,\big((1 + q^2)\,r - 2pqs + (1 + p^2)\,t\big) + k^4 = 0.$$

The two roots of this equation give the greatest and least radii of curvature of the normal sections passing through the point x, y, z.

Example. Find the radii of curvature at any point of a paraboloid.

Let $\dfrac{x^2}{a} + \dfrac{y^2}{b} = 2z$ be the equation of the paraboloid, then $p =$

$\dfrac{dz}{dx} = \dfrac{x}{a}$, $q = \dfrac{dz}{dy} = \dfrac{y}{b}$, and $1 + p^2 + q^2 = k^2 = 1 + \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$;

$r = \dfrac{d^2 z}{dx^2} = \dfrac{1}{a}$, $s = \dfrac{d^2 z}{dx\,dy} = 0$, and $t = \dfrac{d^2 z}{dy^2} = \dfrac{1}{b}$;

$\therefore \varrho^2 - k\,(a + b + 2z)\,\varrho + ab\,k^4 = 0$; and $\varrho =$

$$\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + 1}\left(\frac{a + b + 2z}{2} \pm \sqrt{\left(\frac{a + b + 2z}{2}\right)^2 - ab\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + 1\right)}\right),$$

which gives the radii of minimum and maximum curvature at the point x, y, z.

(161.) To determine the radius of spherical curvature of a curve of double curvature.

The general equation of a sphere is

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=\varrho^2;$$

and in this case y and z are functions of x. If, therefore,

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \text{\&c.} = p', p'', \text{\&c., and } \frac{dz}{dx}, \frac{d^2z}{dx^2} = q', q'', \text{\&c.,}$$

$$x-\alpha+(y-\beta)p'+(z-\gamma)q'=0;$$

$$(y-\beta)p''+(z-\gamma)q''+1+p'^2+q'^2=0;$$

$$(y-\beta)p'''+(z-\gamma)q'''+3(p'p''+q'q'')=0.$$

These four equations enable us to eliminate α, β, γ, and ϱ, and thereby to determine the centre and radius of curvature.

If the proposed point be taken as the origin of co-ordinates, and the tangent at that point as the axis of x, then $x=y=z=0$, and $p'=q'=0$; $\therefore\ \alpha=0$, $\beta p''+\gamma q''=1$, and $\beta p'''+\gamma q'''=0$;

$$\therefore\ \beta=\frac{q'''}{p''q'''-p'''q''}, \text{ and } \gamma=\frac{p'''}{q''p'''-q'''p''};$$

$$\therefore\ \varrho=\frac{\sqrt{(p''')^2+(q''')^2}}{p''q'''-p'''q''}=\frac{\sqrt{\left(\frac{d^3y}{dx^3}\right)^2+\left(\frac{d^3z}{dx^3}\right)^2}}{\frac{d^2y}{dx^2}\frac{d^3z}{dx^3}-\frac{d^3y}{dx^3}\frac{d^2z}{dx^2}}.$$

(162.) To find the equation to the plane which osculates a curve of double curvature at any point.

$x'-x+(y'-y)p'+(z'-z)q'=0$ is the equation to the normal plane through the point x, y, z; and if we differentiate this with regard to x, we have $(y'-y)p''+(z'-z)q''-(1+p'^2+q'^2)=0$, which is the equation to the consecutive normal. But these two normals intersect, and their common section must be a straight line, the equations of whose projections on the planes of xz and yz are

$$x'-x=(z'-z)\frac{q'p''-q''p'}{p''}-(1+p'^2+q'^2)\frac{p'}{p''},$$

$$\text{and } y' - y = -(z' - z)\frac{q''}{p''} + \frac{1 + p'^2 + q'^2}{p''}.$$

Let the equation to the osculating plane be

$$z' - z = A(x' - x) + B(y' - y),$$

then, since this plane must be perpendicular to the line of intersection of the consecutive normals, we have

$$A = -\frac{q''p' - q'p''}{p''}, \text{ and } B = \frac{q''}{p''};$$

$$\therefore z' - z = -(x' - x)\frac{q''p' - q'p''}{p''} + (y' - y)\frac{q''}{p''}.$$

(163.) To find the radius of absolute curvature at any point of a curve of double curvature.

The equation to the osculating plane is

$$z' - z = -(x' - x)\frac{q''p' - q'p''}{p''} + (y' - y)\frac{q''}{p''},$$

which becomes $z' - z = -a(x' - x) + b(y' - y)$ if $a = \frac{q''p' - q'p''}{p''}$,

and $b = \frac{q''}{p''}$; and the equations to the line of intersection of two consecutive normals are

$$x' - x = a(z' - z) - \frac{mp'}{p''}, \text{ and } y' - y = -b(z' - z) + \frac{m}{p''},$$

m being $= 1 + p'^2 + q'^2$, and at the point where this line intersects the osculating plane, we have

$$z' - z = -a^2(z' - z) + \frac{map'}{p''} - b^2(z' - z) + \frac{mb}{p''};$$

$\therefore z' - z = \frac{m}{p''}\,\frac{ap' + b}{1 + a^2 + b^2}$. In a similar manner we have

$$x' - x = -\frac{m}{p''}\cdot\frac{p' + bq'}{1 + a^2 + b^2}$$

and $$y' - y = -\frac{m}{p''}\cdot\frac{aq' - 1}{1 + a^2 + b^2}.$$

But $\rho^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2$;

$$\therefore \rho^2 = \left(\frac{m}{p''}\right)^2 \frac{(ap' + b)^2 + (p' + bq')^2 + (aq' - 1)^2}{(1 + a^2 + b^2)^2}.$$

But since $a = \dfrac{q''p' - q'p''}{p''}$, and $b = \dfrac{q''}{p''}$, we have

$$a + q' - bp' = 0;\ \therefore\ 2(abp' + bp'q' - aq') = a^2 + q'^2 + b^2p'^2;$$

$$\therefore \rho^2 = \left(\frac{m}{p''}\right)^2 \frac{1 + p'^2 + q'^2}{1 + a^2 + b^2} = \frac{(1 + p'^2 + q'^2)^3}{p''^2 + q''^2 + (q''p' - q'p'')^2};$$

$$\therefore \rho = \frac{(1 + p'^2 + q'^2)^{\frac{3}{2}}}{\sqrt{p''^2 + q''^2 + (q''p' - q'p'')^2}};$$ and if the arc s be made the independent variable, we shall have

$$\frac{1}{\rho} = \sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2}.$$

EXAMPLE. To find the radius of absolute curvature of the helix, whose equations are $x = a\cos.\dfrac{z}{h}$, and $y = a\sin.\dfrac{z}{h}$.

Since $a\cos.\dfrac{z}{h} = x$, $z = h\cos.^{-1}\dfrac{x}{a}$, and $dz = -\dfrac{h\,dx}{\sqrt{a^2 - x^2}}$.

Again $a^2\cos.^2\dfrac{z}{h} = x^2$,

$$a^2\sin.^2\frac{z}{h} = y^2;\ \therefore\ a^2 = x^2 + y^2, \text{ and } dy = -\frac{x\,dx}{\sqrt{a^2 - x^2}};$$

$\therefore\ dx^2 + dy^2 + dz^2 = \dfrac{(a^2 + h^2)dx^2}{a^2 - x^2}$, $\therefore\ ds = \sqrt{dx^2 + dy^2 + dz^2}$.

(152) $= \dfrac{(a^2 + h^2)^{\frac{1}{2}}\, dx}{(a^2 - x^2)^{\frac{1}{2}}}$. But since $x = a \cos. \dfrac{z}{h}$, $\dfrac{dx}{ds} = -\dfrac{a}{h} \sin. \dfrac{z}{h}$,

$\times\ \dfrac{dz}{ds} = -\dfrac{y}{\sqrt{a^2 + h^2}}$, $\dfrac{dy}{ds} = \dfrac{x}{\sqrt{a^2 + h^2}}$, and $\dfrac{dz}{ds} = \dfrac{h}{\sqrt{a^2 + h^2}}$,

$\dfrac{d^2x}{ds^2} = -\dfrac{1}{\sqrt{a^2 + h^2}}\dfrac{dy}{ds} = -\dfrac{x}{a^2 + h^2}$, $\dfrac{d^2y}{ds^2} = \dfrac{1}{\sqrt{a^2 + h^2}}\dfrac{dx}{ds} = -\dfrac{y}{a^2 + h^2}$,

and $\dfrac{d^2z}{ds^2} = 0$; $\therefore\ \dfrac{1}{\varrho} = \sqrt{\left(\dfrac{d^2x}{ds^2}\right)^2 + \left(\dfrac{d^2y}{ds^2}\right)^2 + \left(\dfrac{d^2z}{ds^2}\right)^2} = \dfrac{a}{a^2 + h^2}$,

$\therefore\ \varrho = \dfrac{a^2 + h^2}{a}$.

CHAPTER XVII.

CYLINDRICAL, CONICAL, AND CONOIDAL SURFACES, AND SURFACES OF REVOLUTION.

(164.) If a straight line move parallel to itself, and describe with its extremity a given curve, it shall generate a cylindrical surface.

The given line is called the *generatrix*, and the given curve the *directrix*.

(165.) To find the general and differential equations to a cylindrical surface.

Let $x = az + \alpha$, $\therefore \alpha = x - az$,

$y = bz + \beta$; $\therefore \beta = y - bz$, be the equations of the generatrix in any of its positions.

Then since the generatrix always moves parallel to itself, a and b do not vary. But α and β, which are the co-ordinates of the point where the generatrix meets the plane of xy, are constant for the same position of it, and vary as it passes from one point to another; and as they always vary and are constant together, it is obvious that the one must be a function of the other. $\therefore \beta = \varphi(\alpha)$, that is, $y - bz = \varphi(x - az)$, which is the general equation to cylindrical surfaces.

If we eliminate the function from this equation by differentiation, we shall have

$a\dfrac{dz}{dx} - b\dfrac{dz}{dy} = 1$, which is the general differential equation to cylindrical surfaces.

(166.) Given the equations of the generatrix to determine that of a cylindrical surface which shall envelope a given surface.

We have just found the general differential equation of a cylindrical surface to be

$a\frac{dz}{dx} + b\frac{dz}{dy} = 1.$ (1) But at the points where the cylindrical surface is touched by the given surface, the co-ordinates x, y, z must be the same for both; if, therefore, the differential coefficients $\frac{dz}{dx}$ and $\frac{dz}{dy}$ be derived from the equation to the given surface, and substituted in equation (1), they shall fulfil its conditions; and since we have now the result, the equations to the generatrix, and the equation to the given surface, we can determine the equation to the required cylindrical surface.

EXAMPLE. To determine a cylindrical surface which shall envelope a given ellipsoid.

Let the equation to the ellipsoid be

$$A x^2 + B y^2 + C z^2 = 1. \quad (1) \therefore \frac{dz}{dx} = -\frac{A x}{C z}, \text{ and } \frac{dz}{dy} = -\frac{B y}{C z}.$$

Substituting these in the equation $a\frac{dz}{dx} + b\frac{dz}{dy} = 1$, then $A ax + B by + C z = 0$. (2) And the equations to the generatrix are $x = az + \alpha$, and $y = bz + \beta$. (3)

Eliminating x, y, z by means of (1), (2), (3), we have

$$(A\alpha^2 + B\beta^2 - 1)(A a^2 + B b^2 + C) = (A a \alpha + B b \beta)^2.$$

Substituting the values of α and β from (3), we have

$(A(x - az)^2 + B(y - bz)^2 - 1)(A a^2 + B b^2 + C) = (A a(x - az) +$

$B b (y - bz))^2 = ((A ax + B by + Cz) - (A a^2 + B b^2 + C) z)^2$;

$\therefore (A^2 x^2 + B y^2 + C z^2 - 1) \times (A a^2 + B b^2 + C) = (A ax + B by + C z)^2$, which is the equation to the required cylindrical surface.

If the surface be perpendicular to the plane of xy, $a = 0$, and $b = 0$; $\therefore$ the equation becomes $A x^2 + B y^2 = 1$, which is that of an upright elliptical cylinder.

(167.) If a straight line pass constantly through a given point, and describe with its extremity a given curve, it shall generate a conical surface.

The line is called the *generatrix*, the point the *vertex*, and the curve the *directrix*.

(168.) To find the general and differential equations to a conical surface.

Let the co-ordinates of the vertex be a, b, c, then the equations to the generatrix are

$$x - a = \alpha (z - c), \; y - b = \beta (z - c).$$

But when a point on the surface changes its position without leaving the generatrix, α and β are constant, but when it passes from one position of the generatrix to another, α and β both vary. Now, since these quantities both vary, and are both constant together, the one must be a function of the other; $\therefore \beta = \varphi (\alpha)$: that is, $\frac{y - b}{z - c} = \varphi \left(\frac{x - a}{z - c}\right)$, which is the general equation to conical surfaces.

If we eliminate the function from this equation by differentiation, we shall have

$z - c = \frac{dz}{dx} (x - a) + \frac{dz}{dy} (y - b)$, which is the differential equation to conical surfaces.

COR. If the vertex be the origin of co-ordinates, $a = 0$, $b = 0$, $c = 0$, and the above differential equation becomes

$$z = \frac{dz}{dx} x + \frac{dz}{dy} y.$$

(169.) Given the equations to the generatrix to determine that of a conical surface which shall envelope a given surface.

We have just found the differential equation of a conical surface to be

$$z - c = \frac{dz}{dx}(x - a) + \frac{dz}{dy}(y - b). \qquad (1)$$

But at the points where the conical surface is touched by the given surface, the co-ordinates x, y, z must be the same for both. If, therefore, the values of $\frac{dz}{dx}$ and $\frac{dz}{dy}$, derived from the equation to the given surface, be substituted in equation (1), they must fulfil its conditions; and since we have now the result, the equations to the generatrix and the equation to the given surface, we can determine that of the required conical surface.

EXAMPLE. To determine a conical surface which shall envelope a given ellipsoid.

Let the equation to the ellipsoid be

$Ax^2 + By^2 + Cz^2 = 1$ (1), and let the axis of z pass through the vertex of the cone, then $a = 0$ and $b = 0$; $\therefore$ the equation to the conical surface becomes $z - c = \frac{dz}{dx} x + \frac{dz}{dy} y$. But $\frac{dz}{dx} = -\frac{Ax}{Cz}$,

$$\frac{dz}{dy} = -\frac{By}{Cz}; \quad \therefore 1 = Ccz. \qquad (2)$$

Let the equations to the generatrix be $x = \alpha(z - c)$, $y = \beta(z - c)$. (3)

Eliminating x, y, z by means of (1), (2), (3), we have $(A\alpha^2 + B\beta^2)$

$\left(c^2 - \frac{1}{C}\right) = 1$. Substituting for α and β their values from (3), we have $(z-c)^2 = (Ax^2 + By^2)\left(\left(c^2 - \frac{1}{C}\right)\right)$, which is the equation to the required conical surface.

(170.) If a straight line move always parallel to the plane of $x\,y$, and one of its extremities move along the axis of z, while the other describes a given curve, it will generate a conoidal surface.

(171.) To find the general and differential equations to a conoidal surface.

It is evident that $z = \beta$ and $y = \alpha x$ are the equations to the generatrix. Now if a point change its position without leaving the generatrix, α and β are both constant; but if it pass from one position of the generatrix to another, α and β both vary; $\therefore$ since these quantities both vary and are constant together, $\beta = \varphi(\alpha)$: that is, $z = \varphi\left(\frac{y}{x}\right)$, which is the general equation to conoidal surfaces.

If we eliminate the function from this equation by differentiation, we shall have $x\frac{dz}{dx} + y\frac{dz}{dy} = 0$, which is the differential equation to conoidal surfaces.

(172.) If a circle move along a straight line, which passes through its centre, and is perpendicular to its plane, and if the circumference always pass through a given curve, it shall generate a solid of revolution.

The given line is called the axis, and the given curve the directrix.

(173.) To find the general and differential equations to surfaces of revolution.

Let $x = az + \alpha$, and $y = bz + \beta$ be the equations to the axis, then

$ax + by + z = c$, and $(x-\alpha)^2 + (y-\beta)^2 + z^2 = r^2$, must be the equations to the circle.

Now, if any point in the surface change its position without leaving the circumference of the same circle, c and r^2 are both constant; but if it pass from one circle to another, c and r^2 both vary; then, since these quantities both vary and are constant together, $c = \varphi(r^2)$: that is, $ax + by + z = \varphi((x-\alpha)^2 + (y-\beta)^2 + z^2)$, (1) which is the general equation to a surface of revolution.

If the axis of revolution coincide with the axis of z, $a = 0$, $b = 0$, $\alpha = 0$, $\beta = 0$; $\therefore z = \varphi(x^2 + y^2 + z^2)$ or $z = \psi(x^2 + y^2)$.

If we eliminate the function from (1) by differentiation, we shall have $(bz + \beta - y)\dfrac{dz}{dx} - (az + \alpha - x)\dfrac{dz}{dy} + (\beta - y)a - (\alpha - x)b = 0$,

which is the differential equation to a surface of revolution.

(174.) If a surface be generated by the consecutive intersections of a series of planes drawn according to a given law, it shall be a developable surface; that is, one that may be made to coincide with a plane, without tearing or rumpling.

(175.) To find the general and differential equations to a developable surface.

Let $z = Ax + By + C$ be the equation to a plane, then it is necessary, in order that there may be any number of consecutive planes drawn after a given law, that the constants A, B, and C must be functions of the same parameter α;

$$\therefore z = xf(\alpha) + y\,F(\alpha) + \varphi(\alpha). \qquad (1)$$

Then suppose x, y, z to remain constant while α varies, and we have

$$xf'(\alpha) + y\,F'(\alpha) + \varphi'(\alpha) = 0. \qquad (2)$$

$\therefore$ (1) and (2) are the equations to the intersection of two consecutive planes; and if α be eliminated, the result shall be the general equation to developable surfaces.

Again, since $z = xf(\alpha) + y\,F(\alpha) + \phi(\alpha)$,

$$\frac{dz}{dx} = f(\alpha) + (xf'(\alpha) + y\,F'(\alpha) + \phi'(\alpha))\frac{d\alpha}{dx} = f(\alpha) \text{ by } (2);$$

$$\text{and } \frac{dz}{dy} = F(\alpha) + (yF'(\alpha) + xf'(\alpha) + \phi'(\alpha))\frac{d\alpha}{dy} = F(\alpha) \text{ by } (2);$$

$$\therefore \frac{dz}{dx} = \psi\left(\frac{dz}{dy}\right); \therefore \frac{d^2z}{dx^2} = r = \psi'\left(\frac{dz}{dy}\right)\frac{d^2z}{dx\,dy} = \psi'\left(\frac{dz}{dy}\right)s;$$

$\therefore \dfrac{r}{s} = \psi'\left(\dfrac{dz}{dy}\right)$. In a similar manner it appears that $\dfrac{s}{t} = \psi'\left(\dfrac{dz}{dy}\right)$; $\therefore rt - s^2 = 0$, which is the differential equation to developable surfaces.

Examples for Practice on the Three preceding Chapters.

(1.) If $x^2 + y^2 + z^2 = a^2$ be the equation to a sphere, then $xx' + yy' + zz' = a^2$ is the equation to its tangent plane.

(2.) If $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$ be the equation to a hyperboloid of one sheet, then $\dfrac{xx'}{a^2} + \dfrac{yy'}{b^2} - \dfrac{zz'}{c^2} = 1$ is the equation to its tangent plane.

(3.) If $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ be the equation to an ellipsoid, then the pyramid formed by the tangent plane and the three co-ordinate planes $= \dfrac{1}{6}\,\dfrac{a^2 b^2 c^2}{xyz}$.

(4.) If three tangent planes to an ellipsoid whose equation is

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ be at right angles to each other, then their point of intersection will trace out a sphere concentric with the ellipsoid, and whose radius is $\varrho = \sqrt{a^2 + b^2 + c^2}$.

(5.) If $\sqrt{x} + \sqrt{y} + \sqrt{z} = a$ be the equation to a surface; then the sum of the intercepts on the co-ordinate axes made by a tangent plane is constant.

(6.) If $x^2 + y^2 = az$ be the equation to the common paraboloid, then $x' - x + \frac{2x}{a}(z' - z) = 0$, and $y' - y + \frac{2y}{a}(z' - z) = 0$ are the equations to its normal, and $d = z\sqrt{1 + \frac{4z}{a}}$, d being the length intercepted between the surface and the plane of xy.

(7.) If the equations to a curve of double curvature be $x^2 + z^2 = a^2$, and $y^2 + z^2 = b^2$, then the equations to its tangent line are $xx' + zz' = a^2$, and $yy' + zz' = b^2$, and the equation to its normal plane is

$$\frac{x'}{x} + \frac{y'}{y} - \frac{z'}{z} = 1.$$

(8.) The equation to an elliptic paraboloid is $z = \frac{x^2}{a} + \frac{y^2}{b}$, prove that the radius of curvature of a section through the axis of z, which makes an angle of $30°$ with the plane of xz, is $\varrho = \frac{2ab}{a + 3b}$.

(9.) If $z = \frac{x^2}{a} + \frac{y^2}{b}$ be the equation to an elliptic paraboloid, then the radii of curvature of the normal sections of greatest and least curvature passing through the vertex are $\varrho_1 = \frac{b}{2}$, and $\varrho_2 = \frac{a}{2}$.

(10.) In the hyperbolic paraboloid, whose equation is $z = \frac{x^2}{a} - \frac{y^2}{b}$,

prove that the normal sections of greatest and least curvature passing through the vertex are at right angles to each other.

(11.) The equation to the *heliçoide gauche* is $z = \frac{1}{n} \tan.^{-1} \frac{x}{y}$,

prove that $n^2 \rho^2 - (1 + n^2(x^2 + y^2))^2 = 0$ gives the radii of maximum and minimum curvature of the normal sections passing through x, y, z.

(12.) The equation to the equable spherical spiral is $x^2 + y^2 + z^2 = 4r^2$, and $x^2 + y^2 = 2rx$, prove that the radius of absolute curvature

is $\rho = \frac{(2r + x)^{\frac{3}{2}}}{(10r + 3x)^{\frac{1}{2}}}$.

CHAPTER XVIII.

ON THE METHODS OF NEWTON, LEIBNITZ, AND LAGRANGE.

FIRST. NEWTON'S METHOD, OR THAT OF FLUXIONS.

(176.) NEWTON conceived all quantities to be produced by continuous motion. Thus, solids are generated by the motion of surfaces, surfaces by that of lines, and lines by that of points.

(177.) The increment or decrement of a quantity at any instant of time, taken proportional to the velocity with which the quantity flows at that time, Newton called the *Fluxion*, and the quantity itself the *Fluent*.

(178.) The fluxions of the quantities z and x be represented by $\dot{z}$ and $\dot{x}$, $\frac{\dot{z}}{\dot{x}}$ being the ratio between the rates of increase or decrease of the quantities z and x at any point of time is equivalent to $\frac{dz}{dx}$. In like manner $\frac{\ddot{z}}{\dot{x}^2}$, $\frac{\dddot{z}}{\dot{x}^3}$, $\frac{\ddddot{z}}{\dot{x}^4}$, &c. are equivalent to $\frac{d^2z}{dx^2}$, $\frac{d^3z}{dx^3}$, $\frac{d^4z}{dx^4}$, &c., and thus Taylor's Theorem, which by the common notation is $z' = z + \frac{dz}{dx}h + \frac{d^2z}{dx^2}\frac{h^2}{1.2} + \frac{d^3z}{dx^3}\frac{h^3}{1.2.3} +$ &c. will assume the form of

$$z' = z + \frac{\dot{z}}{\dot{x}}h + \frac{\ddot{z}}{\dot{x}^2}\frac{h^2}{1.2} + \frac{\dddot{z}}{\dot{x}^3}\frac{h^3}{1.2.3} + \text{\&c.}$$ when the fluxionary notation is employed.

(179.) If the velocities by which two lines, surfaces, or solids are

generated be uniform, and as $m : n$, the corresponding increments of the quantities will be in the same ratio, which will therefore be the ratio of the fluxions; but if the velocities continually vary, the limit of the ratio is taken as the ratio of the fluxions.*

(180.) We shall now proceed to demonstrate some of the fundamental propositions of the Calculus by the method of fluxions.

(1.) Let $z = x^2$, then $\dot z = 2x\dot x$.

For let τ = the indefinitely small portion of time during which the velocities are continued, and $\dot z$ and $\dot x$ the velocities, then

$z + \tau\dot z = (x = \tau\dot x)^2 = x^2 + 2\tau x\dot x + \tau^2\dot x^2$. But $z = x^2$; ∴ $\tau\dot z = 2\tau x\dot x + \tau^2\dot x^2$; and ∴ $\dot z = 2x\dot x + \tau\dot x^2 = 2x\dot x$, since τ is evanescent.

(2.) Let $z = x^n$, then $\dot z = n x^{n-1}\dot x$.

For $z + \tau\dot z = (x + \tau\dot x)^n = x^n + n\tau x^{n-1}\dot x + \frac{n\overline{n-1}}{1 . 2}\tau^2 x^{n-2}\dot x^2$ + &c. But $z = x^n$; ∴ $\dot z = n x^{n-1}\dot x + \frac{n\overline{n-1}}{1 . 2}\tau x^{n-2}\dot x^2$ + &c. = $n x^{n-1}\dot x$, since τ is ultimately = 0.

(3.) Let $z = xy$, then $\dot z = y\dot x + x\dot y$.

For $z + \tau\dot z = (x + \tau\dot x)(y + \tau\dot y) = xy + \tau y\dot x + \tau x\dot y + \tau^2\dot x\dot y$. But $z = xy$; ∴ $\dot z = y\dot x + x\dot y + \tau\dot x\dot y = y\dot x + x\dot y$, since $\tau = 0$.

(4.) Let $z = a^x$, then $\dot z = A\, a^x\dot x = \log.\, a\, a^x\dot x$.

For $z + \tau\dot z = a^{x + \tau\dot x} = a^x(1 + A\tau\dot x + A^2\tau^2\dot x^2 + \&c.)$ (24) $= a^x + A\, a^x\tau\dot x + A^2 a^x\tau^2\dot x^2$ + &c. But $z = a^x$; ∴ $\dot z = A\, a^x\dot x + A^2\tau\dot x^2$ + &c. $= A\, a^x\dot x$, since τ is of evanescent magnitude; ∴ $\dot z$ = log. $a\, a^x\dot x$. (26)

* If the portions of time, however, during which the motion is continued be taken indefinitely small, the second case will include the first. For when two variable quantities are always in a constant ratio, their limits are in that ratio.

(5.) Let $z = \log. x$, then $\dot{z} = \frac{\dot{x}}{x}$.

For since $\log x = z$, $x = e^z$; $\therefore \dot{x} = A\, e^z\, \dot{z} = e^z\, \dot{z}$, since $A = 1$ in this case (26); $\therefore \dot{z} = \frac{\dot{x}}{x}$.

(6.) Let $z = \sin. x$, then $\dot{z} = \cos. x\, \dot{x}$.

For $z + \tau \dot{z} = \sin. (x + \tau \dot{x}) = \sin. x + 2 \cos. \left(x + \frac{\tau \dot{x}}{2}\right)$ $\sin. \frac{\tau \dot{x}}{2}$. But $z = \sin. x$, $\sin. \frac{\tau \dot{x}}{2} = \frac{\tau \dot{x}}{2}$, and $\cos. \left(x + \frac{\tau \dot{x}}{2}\right) =$ $\cos. x$; $\therefore \dot{z} = \cos. x\, \dot{x}$.

In a similar manner might all the fundamental propositions of the Differential Calculus be demonstrated by the method of fluxions.

SECOND. THE METHOD OF LEIBNITZ, OR THAT OF INFINITESIMALS.

(181.) If a quantity be infinitely great, it cannot be increased by the addition of any finite quantity.

Let x be an infinitely great quantity and a a finite quantity, then $x + a = x$.

For, let $\frac{1}{x} + \frac{1}{a} = M$,

then $x + a = a x M$. But $\frac{1}{x} = \frac{1}{\infty} = 0$; $\therefore M = \frac{1}{a}$,

and $\therefore x + a = x$.

COR. If a be a finite quantity, and x infinitely small compared with a, then $x + a = a$, that is, a finite quantity is not increased by the addition of an infinitely small quantity. The infinitely small quantity x is called an *infinitesimal of the first degree.*

(182.) If x be infinitely small compared with 1, x^2 will be infinitely small compared with x.

For $1 : x :: x : x^2$; $\therefore$ x contains x^2 as often as 1 contains x, that is an infinite number of times. In a similar manner it appears that x^2 contains x^3 as often as 1 contains x, and so on.

$x^2, x^3, x^4 \ldots x^n$ are called infinitesimals of the *second, third, fourth, ... n^{th} degrees.*

(183.) When two infinitesimals of the first degree are multiplied together, their product will be an infinitesimal of the second degree.

Thus, if x and y be each an infinitesimal of the first degree, xy will be an infinitesimal of the second degree.

For $1 : x :: y : xy$, $\therefore$ y contains xy as often as 1 contains x, that is, an infinite number of times.

In a similar manner it appears that the product of three infinitesimals of the first degree is an infinitesimal of the third degree, and so on.

(184.) If x be an infinitely small quantity, and m any finite quantity, then mx is an infinitely small quantity.

For let $x = \frac{a}{\infty}$, then $m\,x = \frac{m\,a}{\infty}$ = an infinitely small quantity.

(185.) The ratio of two infinitesimals of the same degree is a finite quantity.

Let x and y be two infinitesimals of the same degree, then $\frac{x}{y}$ = a finite quantity.

For let $x = \frac{a}{\infty}$, and $y = \frac{b}{\infty}$, then $\frac{x}{y} = \frac{\frac{a}{\infty}}{\frac{b}{\infty}} = \frac{a}{b}$ = a finite quantity.

(186.) We shall now proceed to apply these principles to the demonstration of some propositions in the Differential Calculus.

(1.) Let $z = x^3$, then $\frac{dz}{dx} = 3x^2$.

For let x become $= x + dx$, and $z = z + dz$, where dx and dz are infinitesimals of the first degree, then

$z + dz = (x + dx)^3 = x^3 + 3\,x^2\,dx + 3\,x\,dx^2 + dx^3$. But $z = x^3$; $\therefore\ dz = 3\,x^2\,dx + 3\,x\,dx^2 + dx^3$. Now dx^3 being an infinitesimal of the third degree $= 0$, compared to $3x\,dx^2$. For the same reason $3x\,dx^2 = 0$, compared with $3x^2\,dx$. Hence dx^3 and $3\,x\,dx^2$ may be omitted. $\therefore\ dz = 3\,x^2\,dx$, and $\frac{dz}{dx} = 3\,x^2$.

(2.) Let $z = x^n$, then $\frac{dz}{dx} = nx^{n-1}$.

For $z + dz = (x + dx)^n = x^n + nx^{n-1}\,dx + \frac{n\overline{n-1}}{1\,.\,2}\,x^{n-2}\,dx^2 + \ldots$

But $z = x^n$; $\therefore\ dz = nx^{n-1}\,dx + \frac{n\overline{n-1}}{1\,.\,2}\,x^{n-2}\,dx^2 + \ldots$

Now, dx being an infinitesimal of the first degree dx^2, dx^3, dx^4, ... dx^n will be infinitesimals of the second, third, ... nth degrees, and therefore the terms involving them are $= 0$, compared to $nx^{n-1}\,dx$;

$$\therefore\ \frac{dz}{dx} = nx^{n-1}.$$

(3.) Let $z = xy$, then $dz = y\,dx + x\,dy$.

For $z + dz = (x + dx)(y + dy) = xy + y\,dx + x\,dy + dx\,dy$. But $z = xy$; $\therefore\ dz = y\,dx + x\,dy + dx\,dy$. But $dx\,dy$ being the product of two infinitesimals of the first degree, is an infinitesimal of the second degree (138), and $\therefore = 0$, compared to dx or dy; $\therefore\ dz = y\,dx + x\,dy$.

(4.) Let $z = \sin. x$, then $dz = \cos. x\,dx$.

For $z + dz = \sin.(x + dx) = \sin. x \cos. dx + \cos. x \sin. dx$. But $\cos. dx = 1$, and $\sin. dx = dx$; $\therefore\ z + dz = \sin. x + \cos. x\,dx$; $\therefore\ dz = \cos. x\,dx$.

(5.) To find the subtangent AB (fig. page 96) by the method of infinitesimals.

Let OB $= x$, BC $= y$, BF $= dx$, and EG $= dy$, then EG : GC :: CB : BD —that is, $dy : dx :: y :$ BD. But since dx and dy are infinitesimals, the point E must coincide with C, and D with A ; $\therefore dy : dx :: y :$ AB ; $\therefore$ AB $= y \dfrac{dx}{dy}$.

(6.) Let s be the arc of a curve (fig. page 109) AN $= x$, PN $= y$, NN$'$ $= dx$, and P$'$Q $= dy$, then PP$'$, which is represented by ds, being an infinitely small portion of the arc, is a straight line ; $\therefore$ PP$'^2$ $=$ PQ^2 $+$ QP$'^2$; that is, $ds = \sqrt{dx^2 + dy^2}$.

In a similar manner may all the other propositions of the Differential Calculus be demonstrated by the method of infinitesimals.

THIRD. THE METHOD OF LAGRANGE, OR THAT OF DERIVED FUNCTIONS.

(187.) In all the methods which we have as yet employed for demonstrating the rules of the Differential Calculus, there is a certain metaphysical difficulty which is not easily overcome. This Lagrange obviated in the following manner. He proceeded to demonstrate Taylor's Theorem by the aid of common algebra alone, and then to deduce the principles of differentiation from it. In this way he was enabled to dispense with every consideration of limits, infinitesimals, and evanescent quantities.

Lagrange's method is nearly as follows :—

(188.) If $f(x)$ represent a function of any variable quantity x, and if $x + h$ be substituted for x, h being any indeterminate quantity, $f(x)$ will become $f(x + h)$, which may be developed in the form

$$f(x + h) = f(x) + ph + qh^2 + rh^3 + sh^4 + th^5 + \text{\&c.}$$

(1.) In this development none of the exponents of h can be fractional, for, if so, let the series be

$$f(x + h) = f(x) + ph + qh^2 + \ . \ . \ . + uh^{\frac{m}{n}} + \ . \ . \ .$$

Now, since x and h are both indeterminate, $f(x)$ must have as many

values as $f(x+h)$, and $\therefore$ the sum of the terms of the series after $f(x)$, viz. $ph + qh^2 + \ldots + uh^{\frac{m}{n}} + \ldots$ can have only one value. But $uh^{\frac{m}{n}}$ has as many different values as there are units in n, and each value of $f(x)$ will combine with each of these values, so that $f(x+h)$, when developed, will have more values than when not developed, which is absurd.

This demonstration is general and rigid so long as x and h are both indeterminate; but it is possible that particular values given to x may destroy some of the radicals in $f(x)$, which may nevertheless still exist in $f(x+h)$. This is the particular case in which Taylor's Theorem fails. (64)

(2.) None of the exponents of h can be negative; for if so, let

$$f(x+h) = f(x) + ph + qh^2 + \ldots + uh^{-m} + \ldots$$

Then, when $h = 0$, $f(x+h)$ becomes $f(x)$, and $uh^{-m} = \frac{u}{h^m} = \frac{u}{0} = \infty$; $\therefore f(x) = f(x) + \infty$, which is impossible.

(3.) Since, when $h = 0$, $f(x+h)$ must necessarily become $f(x)$; $\therefore$ the remaining part of the series must be multiplied by a positive power of h, and as we have already demonstrated that there cannot enter into the development a fractional power of h, this power of h must be a positive integer. It will then be of the form $\mathrm{P}h$, where P is a function of x and h, which does not become infinite when $h = 0$.

$$\therefore f(x+h) = f(x) + \mathrm{P}h.$$

But P being a new function of x and h, we can separate from it that part which is independent of h, and which by consequence does not vanish when $h = 0$.

Let p = what P becomes when $h = 0$, then p will be a function of x without h, and $\mathrm{P} = p + \mathrm{Q}h$, $\mathrm{Q}h$ being the part of P which becomes nothing when $h = 0$, and Q a new function of x and h, which does not become infinite when $h = 0$.

In a similar manner it appears that $\mathrm{Q} = q + \mathrm{R}h$, $\mathrm{R} = r + \mathrm{S}h$, and $\mathrm{S} = s + \mathrm{T}h$, &c.

$\therefore f(x+h) = f(x) + \mathrm{P}h = f(x) + ph + \mathrm{Q}h^2 = f(x) + ph + qh^2 + \mathrm{R}h^3 = f(x) + ph + qh^2 + rh^3 + \mathrm{S}h^4 = f(x) + ph + qh^2 + rh^3 + sh^4 + \mathrm{T}h^5 = f(x) + ph + qh^2 + rh^3 + sh^4 + th^5 + \&c.$

(189.) We shall now proceed to investigate a general law for deriving the coefficients p, q, r, &c. from $f(x)$ in the formula $f(x+h) = f(x) + ph + qh^2 + rh^3 + sh^4 + \&c.$

For this purpose let $h+i$ be substituted for h in $f(x+h)$ and its expansion, and we shall have

$$f(x+h+i) = f(x) + p(h+i) + q(h+i)^2 + r(h+i)^3 + \&c.$$

Then, taking only the two first terms of the developments of these binomials, we have

$$\left.\begin{aligned} f(x+h+i) = f(x) + ph + qh^2 + rh^3 + sh^4 + \&c. \\ + pi + 2\,qhi + 3\,rh^2i + 4\,sh^3i + \&c. \end{aligned}\right\}\ (1)$$

Next, let $x+i$ be substituted for x in $f(x+h)$, and its development. Then, since p, q, r, &c. are functions of x without h, by (3) of (188), we have

$$\begin{gathered} f(x+h+i) = f(x) + f'(x)i + \&c. + (p + p'i + \&c.)h + \\ (q + q'i + \&c.)h^2 + (r + r'i + \&c.)h^3 + \&c. = \end{gathered}$$

$$\left.\begin{aligned} f(x) + ph + qh^2 + rh^3 + sh^4 + \&c. \\ + f'(x)i + p'hi + q'h^2i + r'h^3i + \&c. \end{aligned}\right\}\ (2)$$

But the developments (1) and (2) of $f(x+h+i)$ must be identical; $\therefore p = f'(x)$, $2\,q = p'$, $3\,r = q'$, $4\,s = r'$, &c. = &c.

Now if $f'(x), f''(x), f'''(x)$, &c. represent the first, second, third, &c. functions derived from $f(x)$, since p' is derived from p, q' from q, r' from r, &c. in the same manner as $f'(x)$ is derived from $f(x)$, we have $p = f'(x)$, $\therefore p' = f''(x)$, $q = \frac{p'}{1\,.\,2} = \frac{f''(x)}{1\,.\,2}$, $\therefore q' = \frac{f'''(x)}{1\,.\,2}$, $r = \frac{q'}{3} = \frac{f'''(x)}{1\,.\,2\,.\,3}$, $\therefore r' = \frac{f''''(x)}{1\,.\,2\,.\,3}$, $s = \frac{r'}{4} = \frac{f''''(x)}{1\,.\,2\,.\,3\,.\,4}$, &c. = &c.

$$\therefore f(x+h) = f(x) + f'(x)\frac{h}{1} + f''(x)\frac{h^2}{1.2} + f'''(x)\frac{h^3}{1.2.3} + f''''(x)\frac{h^4}{1.2.3.4} + \&c.$$

(190.) If y, y', y'', y''', y'''', &c. represent $f(x), f'(x), f''(x), f'''(x), f''''(x)$, &c. we shall have

$$f(x+h) = y + y'\frac{h}{1} + y''\frac{h^2}{1.2} + y'''\frac{h^3}{1.2.3} + y''''\frac{h^4}{1.2.3.4} + \&c.$$

(191.) Since $f'(x)$ is the coefficient of h in the expansion of $f(x+h)$, it is equal to what we represent by $\frac{df(x)}{dx}$, or by $\frac{dy}{dx}$. For a similar reason, $f''(x) = \frac{d^2y}{dx^2}$, $f'''(x) = \frac{d^3y}{dx^3}$, &c; $\therefore f(x+h) = f(x) + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\frac{h^3}{1.2.3}$ + &c. which is Taylor's Theorem, derived without the aid of the Differential Calculus.

(192.) We shall now proceed to illustrate Lagrange's method by examples.

(1.) Let $z = x^3$, then $\frac{dz}{dx} = 3x^2$.

For substitute $x + h$ for x, and x^3 becomes $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$. But $\frac{dz}{dx}$ is the coefficient of h in the expansion of $f(x+h)$ (191); $\therefore \frac{dz}{dx} = 3x^2$.

(2.) Let $z = ax^n$, then $\frac{dz}{dx} = nax^{n-1}$.

For let $x + h$ be substituted for x, then ax^n becomes $a(x + h)^n$

$$= ax^n + n\,ax^{n-1}h + \frac{n\,\overline{n-1}}{1\,.\,2}\,ax^{n-2}h^2 + \&c.\,; \therefore \frac{dz}{dx} = n\,ax^{n-1}.$$

(3.) Let $z = a^x$, then $\frac{dz}{dx} = \text{A}\,a^x = \log.\,a\,a^x$.

For substitute $x + h$ for x, then a^x becomes $a^{x+h} = a^x + \text{A}\,a^x h +$

$$\text{A}^2\frac{a^x h^2}{1\,.\,2} + \&c.\,; \therefore \frac{dz}{dx} = \text{A}\,a^x = \log.\,a\,a^x.$$

(4.) Let $z = \log.\,x$, then $\frac{dz}{dx} = \frac{1}{x}$.

For let $x + h$ be substituted for x, then $\log\,x$ becomes $\log.\,(x + h)$

$$\log.\,x + \frac{h}{x} - \frac{h^2}{2\,x^2} + \frac{h^3}{3\,x^3} - \&c. \text{ by common algebra}\,; \therefore \frac{dz}{dx} = \frac{1}{x}.$$

(5.) Let $z = \sin.\,x$, then $\frac{dz}{dx} = \cos.\,x$.

For substitute $x + h$ for x, then $\sin.\,x$ becomes $\sin.\,(x + h) =$

$$\sin\,x\cos\,h + \cos.\,x\sin\,h = \sin.\,x\left(1 - \frac{h^2}{1\,.\,2} + \frac{h^4}{1\,.\,2\,.\,3\,.\,4} - \&c.\right)$$

$$+ \cos.\,x\left(h - \frac{h^3}{1\,.\,2\,.\,3} + \frac{h^5}{1\,.\,2\,.\,3\,.\,4\,.\,5} - \&c.\right) = \sin.\,x + \cos.\,x\,h$$

$$- \sin.\,x\,\frac{h^2}{1\,.\,2} - \cos.\,x\,\frac{h^3}{1\,.\,2\,.\,3} + \sin.\,x\,\frac{h^4}{1\,.\,2\,.\,3\,.\,4} + \cos.\,x\,\frac{h^5}{1\,.\,2\,.\,3\,.\,4\,.\,5}$$

$$- \&c.\,, \therefore \frac{dz}{dx} = \cos.\,x.$$

(193.) It is obvious that all that is necessary is to expand the different functions $(x + h)^n$, a^{x+h}, log. $(x + h)$, sin. $(x + h)$, &c. and we have at once, not only the first differential coefficients, but also the

second, third, fourth, &c. from the same expansion. These functions are all numerous, and can all be developed by common algebra, in ascending powers of h, unless when particular values are given to x, but the processes are in some cases so complex, that it is found easier by the student to determine the value of a ratio whose terms are evanescent than to master them. Moreover, although according to Lagrange's method neither limits nor infinitesimals are required for deducing the rules for differentiation, when these rules are applied to various problems, we are still under the necessity of introducing the idea of evanescent quantities.

It appears, therefore, that Lagrange has not entirely obviated the difficulty; and although he has established Taylor's Theorem by the ordinary rules of algebra in a very logical manner, it is better to obtain both it and Maclaurin's by the aid of the Differential Calculus, and then to employ them in the development of functions, as the expansions of many functions are obtained with great facility by their aid, which, by the ordinary algebraic processes, are found to be very intricate.

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